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CONTENTS

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REPRESENTATIONS OF SEMISIMPLE LIE GROUPS, V.*

By Harish-Chandra.

1. Introduction. Let G be a connected semisimple Lie group and A a Cartan subgroup of G. Under the assumption that the image of A in the adjoint group of G is compact, we have studied in detail [5(f)] certain irreducible representations of the Lie algebra of G and seen that they can all be 'extended" to representations of the group (Theorem 4 of [5(f)]). In the present paper we shall obtain them directly as irreducible representations of G on certain Hilbert spaces consisting of holomorphic functions on a suitable complex manifold.

It turns out that these representations are very intimately related with the finite-dimensional ones (Lemma 14) and the two have some striking similarities (Lemmas 6 and 12). Moreover, as we shall see in Theorem 3, under appropriate conditions some of these representations are unitary. In the last section we obtain a result on their characters. Some of the deeper analogies between these representations of G and those of a compact smisimple group (see [5(e)]) will be discussed in another paper.

2. Certain complex manifolds. We keep to the notation of [5(f), § 3]. Let $\mathfrak{g}_0$ be a semisimple Lie algebra over the field R of real numbers. Define $\mathfrak{k}_0$ and $\mathfrak{p}_0$ as in [5(b), § 2] and let $\mathfrak{h}_0$ be a maximal abelian subalgebra of $\mathfrak{k}_0$. *We shall assume that* $\mathfrak{h}_0$ *is also maximal abelian in* $\mathfrak{g}_0$ (see [5(f), § 3]. Let C denote the field of complex numbers. We complexify $\mathfrak{g}_0$, $\mathfrak{h}_0$, $\mathfrak{k}_0$, $\mathfrak{p}_0$ to $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{k}$, $\mathfrak{p}$ respectively. Let $X \to adX$ $(X \varepsilon \mathfrak{g})$ denote the adjoint representation of $\mathfrak{g}$ and θ the automorphism of order 2 given by $\theta(X+Y) = X - Y$ $(X \varepsilon \mathfrak{k}, Y \varepsilon \mathfrak{p})$. Put[1] $\mathfrak{u} = \mathfrak{k}_0 + (-1)^{\frac{1}{2}}\mathfrak{p}_0$. Then $\mathfrak{u}$ is a compact real form of $\mathfrak{g}$. We denote by $\tilde{\theta}$ and η the conjugations of $\mathfrak{g}$ with respect to $\mathfrak{u}$ and $\mathfrak{g}_0$ respectively. Then

$$\tilde{\theta}(X + (-1)^{\frac{1}{2}}Y) = X - (-1)^{\frac{1}{2}}Y \quad (X, Y \varepsilon \mathfrak{u}),$$

$$\eta(X' + (-1)^{\frac{1}{2}}Y') = X' - (-1)^{\frac{1}{2}}Y' \quad (X', Y' \varepsilon \mathfrak{g}_0)$$

and $\eta = \tilde{\theta}\theta$.

We consider an arbitrary but fixed order (see [5(f), § 2]) in the space

* Received July 8, 1955.

[1] We fix once for all a square-root of -1 in C and denote it by $(-1)^{\frac{1}{2}}$.

$\mathfrak{F}_R$ of real linear functions[2] on $\mathfrak{h}$. Put[3] $\mathfrak{n}_+ = \sum_{\alpha>0} CX_\alpha$ and $\mathfrak{n}_- = \sum_{\alpha>0} CX_{-\alpha}$ (where α runs over all positive roots) and let G_c be the simply connected complex analytic group with the Lie algebra $\mathfrak{g}$. We denote by G_0, K_0, A_0, A_+, A_c, N_c^+, N_c^-, U the (real) analytic subgroups of G_c corresponding to $\mathfrak{g}_0$, $\mathfrak{k}_0$, $\mathfrak{h}_0$, $(-1)^{\frac{1}{2}}\mathfrak{h}_0$, $\mathfrak{h}$, $\mathfrak{n}_+$, $\mathfrak{n}_-$, $\mathfrak{u}$ respectively. Then K_0, A_0 and U are compact and U, A_+, N_c^+, N_c^- are simply connected.[4] We now 'extend' θ, $\bar{\theta}$ and η to automorphisms of G_c. Since $\bar{\theta}(\mathfrak{n}_+) = \eta(\mathfrak{n}_+) = \mathfrak{n}_-$, it follows that $\bar{\theta}(N_c^+)$ $\eta(N_c^+) = N_c^-$. According to a theorem of Iwasawa [6] (see also Lemma 26 of [5(b)]) the mapping $(u, h, n) \to uhn$ $(u \in U, h \in A_+, n \in N_c^+)$ is a one-one regular analytic mapping of $U \times A_+ \times N_c^+$ onto G_c. We denote by $z \to Ad(z)$ $(z \in G_c)$ the adjoint representation of G_c.

LEMMA 1. *$N_c^- \cap A_+N_c^+ = \{1\}$ and $N_c^-A_+ \cap N_c^+ = \{1\}$.*

For suppose $an \in N_c^-$ $(a \in A_+, n \in N_c^+)$. Since $\mathfrak{n}_+$ is a nilpotent algebra, it is mapped onto N_c^+ under the exponential mapping (see Birkhoff [1]). Similarly for $\mathfrak{n}_-$. Hence we can choose $X \in \mathfrak{n}_+$ and $Y \in \mathfrak{n}_-$ such that $n = \exp X$, $an = \exp Y$. Now suppose $an \neq 1$. Then $Y \neq 0$ and there exists an element $H \in \mathfrak{h}$ such that $\exp(adY)H = H + Z$ where $Z \in \mathfrak{n}_-$ and $Z \neq 0$ (Lemma 8 of [5(d)]). Therefore it is obvious that

$$H + Z = \exp(adY)H = Ad(an)H = H + Z',$$

where $Z' \in \mathfrak{n}_+$. But since $\mathfrak{n}_+ \cap \mathfrak{n}_- = \{0\}$ it follows that $Z = Z' = 0$ and so we get a contradiction. This proves that $N_c^- \cap A_+N_c^+ = \{1\}$. If we transform this result under the mapping $z \to \bar{\theta}(z^{-1})$ $(z \in G_c)$, we find that $N_c^-A_+ \cap N_c^+ = \{1\}$.

COROLLARY. *$G_0 \cap A_+N_c^+ = \{1\}$ and $N_c^-A_+ \cap G_0 = \{1\}$.*

For suppose $x = an$ $(x \in G_0, a \in A_+, n \in N_c^+)$. Then $an = x = \eta(x) = a^{-1}\eta(n)$, since $\eta(H) = -H$ if $H \in (-1)^{\frac{1}{2}}\mathfrak{h}_0$. Therefore

$$\eta(n) = a^2 n \in N_c^- \cap A_+N_c^+ = \{1\}$$

and so $a^2 = n = 1$. Since A_+ is abelian and simply connected this implies that $a = 1$ and therefore $x = 1$. The second assertion follows from the first under the mapping $z \to \eta(z^{-1})$ $(z \in G_c)$.

---

[2] A linear function on $\mathfrak{h}$ is called real if it takes real values on $(-1)^{\frac{1}{2}}\mathfrak{h}_0$ (see [5(f), § 2]).

[3] Any undefined terms or symbols should automatically be given the same meaning as in [5(f)].

[4] The proofs of these statements are well known and the necessary references can be found in [5(b)]. See in particular [3] and [7].

LEMMA 2. *$G_0A_+N_c^+$ and $N_c^-A_+G_0$ are open in G_c.*

Since $N_c^-A_+G_0$ is the image of $G_0A_+N_c^+$ under the topological mapping $z \to \eta(z^{-1})$ $(z \in G_c)$ it is enough to prove that $G_0A_+N_c^+$ is open. Let $\mathfrak{s}_+ = (-1)^{\frac{1}{2}}\mathfrak{h}_0 + \mathfrak{n}_+$. Then $\mathfrak{s}_+$ is the Lie algebra of $A_+N_c^+$ and so in view of the corollary $\mathfrak{g}_0 \cap \mathfrak{s}_+ = \{0\}$. Hence

$$\dim_R(\mathfrak{g}_0 + \mathfrak{s}_+) = \dim_R \mathfrak{g}_0 + \dim_R \mathfrak{s}_+.$$

On the other hand $\mathfrak{g}$ is the direct sum of $\mathfrak{h}$, $\mathfrak{n}_+$ and $\mathfrak{n}_-$ and therefore

$$\begin{aligned}\dim_R \mathfrak{g}_0 &= \dim_C \mathfrak{g} = \dim_C \mathfrak{h} + \dim_C \mathfrak{n}_+ + \dim_C \mathfrak{n}_- \\ &= \dim_R \mathfrak{h}_0 + 2\dim_C \mathfrak{n}_+ = \dim_R \mathfrak{h}_0 + \dim_R \mathfrak{n}_+ = \dim_R \mathfrak{s}_+.\end{aligned}$$

This shows that $\dim_R(\mathfrak{g}_0 + \mathfrak{s}_+) = 2\dim_R \mathfrak{g}_0 = \dim_R \mathfrak{g}$, and therefore $\mathfrak{g}$ is the direct sum of $\mathfrak{g}_0$ and $\mathfrak{s}_+$. Our assertion now follows from Lemma 26 of [5(b)].

Since A_+ is simply connected, for every $a \in A$ there exists a unique element $H \in (-1)^{\frac{1}{2}}\mathfrak{h}_0$ such that $a = \exp H$. We denote this element by $\log a$. Also any element $z \in G_c$ can be written uniquely in the form $z = uhn$ $(u \in U$, $h \in A_+$, $n \in N_c^+)$. We write $u(z)$ and $H(z)$ to denote u and $\log h$ respectively. Then $z \to u(z)$ and $z \to H(z)$ are (real) analytic mappings of G_c into U and $(-1)^{\frac{1}{2}}\mathfrak{h}_0$ respectively.

LEMMA 3. *Let 2ρ denote the sum of all the positive roots of $\mathfrak{g}$. Then if dx is the Haar measure of G_0,*

$$\int_{G_0} e^{-4\rho(H(x))}dx < \infty.$$

Let ψ denote the mapping $x \to u(x)$ $(x \in G_0)$ of G_0 into U. It follows from the corollary to Lemma 1 that ψ is univalent. Let $x = us$ $(x \in G_0, u \in U$, $s \in A_+N_c^+)$. Then a simple calculation shows that if $X \in \mathfrak{g}_0$, $(d\psi)_x X = V$ where V is the element of $\mathfrak{u}$ determined by the condition $V \equiv Ad(s)X \bmod \mathfrak{s}_+$. (Here $(d\psi)_x$ is the differential of ψ at x (see Chevalley [4]). Now we regard $\mathfrak{g}/\mathfrak{s}_+$ as a vector space over R and denote by D its endomorphism corresponding to $Ad(s)$. Since $\det(Ad(s)) = 1$, $\det D = \det(Ad(s^{-1}))_{\mathfrak{s}_+}$ where $(Ad(s^{-1}))_{\mathfrak{s}_+}$ is the restriction of $Ad(s^{-1})$ on $\mathfrak{s}_+$. Then if $s = an$ $(a \in A_+, n \in N_c^+)$,

$$\det(Ad(s^{-1}))_{\mathfrak{s}_+} = |e^{-2\rho(\log a)}|^2 = e^{-4\rho(\log a)}$$

and therefore $\det D = e^{-4\rho(\log a)} = e^{-4\rho(H(x))}$. This calculation shows that

$$du = e^{-4\rho(H(x))}dx,$$

where du and dx are the Haar measures on U and G_0 respectively. Hence

$$\int_{G_0} e^{-4\rho(H(x))}dx = \int_{\psi(G_0)} du \leqq \int_U du < \infty$$

because U is compact.

Let Q_+ be the set of all totally positive roots of $\mathfrak{g}$ and Q the remaining set of positive roots (see [5(f)]). Put $\mathfrak{p}_+ = \sum_{\beta\varepsilon Q_+} CX_\beta$, $\mathfrak{p}_- = \sum_{\beta\varepsilon Q_+} CX_{-\beta}$, $\mathfrak{p}' = \sum_{\beta\varepsilon Q}(CX_\beta + CX_{-\beta})$ and $\mathfrak{m} = \mathfrak{k} + \mathfrak{p}'$. Then $\mathfrak{p}_+$, $\mathfrak{p}_-$, $\mathfrak{m}$ are all subalgebras of $\mathfrak{g}$ which is a direct sum of these three. Moreover $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are abelian and $[\mathfrak{m}, \mathfrak{p}_+] \subset \mathfrak{p}_+$, $[\mathfrak{m}, \mathfrak{p}_-] \subset \mathfrak{p}_-$ (see [5(f), §§4 and 5] for the proofs of these statements). Let P_c^+, P_c^- and M_c be the complex analytic subgroups of G_c corresponding to $\mathfrak{p}_+$, $\mathfrak{p}_-$ and $\mathfrak{m}$ respectively.

LEMMA 4. *The mapping* $(q, m, p) \rightarrow qmp$ $(q\,\varepsilon\, P_c^-, m\,\varepsilon\, M_c, p\,\varepsilon\, P_c^+)$ *of* $P_c^- \times M_c \times P_c^+$ *into* G_c *is univalent, holomorphic and regular.*

First we claim that $P_c^-M_c \cap P_c^+ = \{1\}$. For suppose $y\,\varepsilon\, P_c^-M_c \cap P_c^+$. Since P_c^+ is abelian, we can choose $Y\,\varepsilon\,\mathfrak{p}_+$ such that $y = \exp Y$. Also since $[\mathfrak{m}, \mathfrak{p}_-] \subset \mathfrak{p}_-$ and since $y\,\varepsilon\, P_c^-M_c$, it is clear that $Ad(y)\mathfrak{p}_- = \mathfrak{p}_-$. Now suppose $y \neq 1$ so that $Y \neq 0$. Then $Y = \sum_\gamma c_\gamma X_\gamma$ where γ runs over all the totally positive roots and $c_\gamma\,\varepsilon\, C$. Let γ_0 be the lowest root such that $c_{\gamma_0} \neq 0$. Then $[Y, X_{-\gamma_0}] \equiv c_{\gamma_0}H_{\gamma_0} \bmod \mathfrak{n}_+$ and therefore it is obvious that

$$Ad(y)X_{-\gamma_0} \equiv X_{-\gamma_0} + c_{\gamma_0}H_{\gamma_0} \bmod \mathfrak{n}_+.$$

However $H_{\gamma_0} \not\varepsilon\, \mathfrak{n}_+ + \mathfrak{n}_-$ and so it follows that $Ad(y)X_{-\gamma_0} \not\varepsilon\, \mathfrak{p}_-$. Since this contradicts the fact that $Ad(y)\mathfrak{p}_- = \mathfrak{p}_-$ we must have $y = 1$ and therefore $P_c^-M_c \cap P_c^+ = \{1\}$. Taking the image of this equality under the mapping $z \rightarrow \eta(z^{-1})$ $(z\,\varepsilon\, G_c)$ we get $P_c^- \cap M_cP_c^+ = \{1\}$. Now if we make use of the fact that both $P_c^-M_c$ and $M_cP_c^+$ are subgroups of G_c, the univalence of our mapping follows straightaway. That it is holomorphic is an immediate consequence of the complex analyticity of G_c. Similarly the regularity follows from the relation $\mathfrak{g} = \mathfrak{p}_- + \mathfrak{m} + \mathfrak{p}_+$ (Lemma 26 of [5(b)]).

It is clear from Lemmas 2 and 4 that $G_0A_+N_c^+$, $N_c^-A_+G_0$ and $P_c^-M_cP_c^+$ are open connected subsets of G_c and therefore they may be regarded as open submanifolds of G_c.

LEMMA 5. $N_c^-A_+G_0A_+N_c^+$ *is contained in* $P_c^-M_cP_c^+$.

Let $\exp \mathfrak{p}_0$ denote the set of all elements in G_0 of the form $\exp X$ $(X\,\varepsilon\,\mathfrak{p}_0)$. Then if $p\,\varepsilon\,\exp \mathfrak{p}_0$ and $p = uan$ $(u\,\varepsilon\, U, a\,\varepsilon\, A_+, n\,\varepsilon\, N_c^+)$,

$$p^{-1} = \theta(p) = ua^{-1}\tilde{\theta}(n)$$

since $\tilde{\theta}(u) = u$ and $\tilde{\theta}(a) = a^{-1}$. Therefore

$$p^2 = \theta(n^{-1}) a^2 n \,\varepsilon\, N_c^- A_+ N_c^+.$$

Now let X be any element in $\mathfrak{p}_0$ and put $p = \exp(\frac{1}{2}X)$. Then

$$p^2 = \exp X \,\varepsilon\, N_c^- A_+ N_c^+$$

and therefore $\exp \mathfrak{p}_0 \subset N_c^- A_+ N_c^+$. On the other hand $\mathfrak{m} + \mathfrak{p}_+ \supset \mathfrak{h} + \mathfrak{n}_+$ and $\mathfrak{p}_- + \mathfrak{m} \supset \mathfrak{n}_- + \mathfrak{h}$. Therefore $A_+ N_c^+ \subset M_c P_c^+$ and $N_c^- A_+ \subset P_c^- M_c$ and so it follows that $\exp \mathfrak{p}_0 \subset P_c^- M_c P_c^+$. But $G_0 = K_0 (\exp \mathfrak{p}_0)$ (see Cartan [3] and Mostow [7]) and $K_0 P_c^- = P_c^- K_0$ since $[\mathfrak{k}, \mathfrak{p}_-] \subset \mathfrak{p}_-$. Moreover $K_0 \subset M_c$ and therefore $G_0 \subset P_c^- M_c P_c^+$. However we have seen already that $A_+ N_c^+ \subset M_c P_c^+$ and $N_c^- A_+ \subset P_c^- M_c$ and so it follows that $N_c^- A_+ G_0 A_+ N_c^+ \subset P_c^- M_c P_c^+$.

3. The simply connected covering manifold of $N_c^- A_+ G_0$. Let S denote the subgroup $N_c^- A_+$ of G_c. Then we have seen that $\bar{W} = S G_0$ is an open submanifold of G_c. Since $\mathfrak{g} = \mathfrak{n}_- + (-1)^{\frac{1}{2}} \mathfrak{h}_0 + \mathfrak{g}_0$ (see the proof of Lemma 2) the maping $(s, \bar{x}) \to s\bar{x}$ $(s \,\varepsilon\, S, \bar{x} \,\varepsilon\, G_0)$ of $S \times G_0$ into $\bar{W}$ is everywhere regular (Lemma 26 of [5(b)]) and therefore open. Also $S \cap G_0 = \{1\}$ (corollary to Lemma 1) and so this mapping is univalent. Hence it is a homeomorphism. Let G be the simply connected covering group of G_0 and let $x \to \bar{x}$ $(x \,\varepsilon\, G)$ denote the natural homomorphism of G onto G_0. Since S is simply connected, $S \times G$ is a simply connected covering space of $S \times G_0$. Therefore we may also regard it as a covering space of $\bar{W}$ under the mapping $\nu: (s, x) \to s\bar{x}$ $(s \,\varepsilon\, S, x \,\varepsilon\, G)$. Since $\bar{W}$ is a complex manifold, we can introduce a complex structure in $S \times G$ in such a way that it becomes a covering manifold of $\bar{W}$ with respect to the mapping ν. Let W denote the complex manifold arising from $S \times G$ in this way. We identify S and G with subsets of W under the topological mappings $s \to (s, 1)$ $(s \,\varepsilon\, S)$ and $x \to (1, x)$ $(x \,\varepsilon\, G)$. Then $S \cap G = (1, 1)$ is the common unit element of S and G which we shall denote by 1. Let $\bar{W}_l$ and $\bar{W}_r$ respectively be the set of all elements $\bar{v}$ and $\bar{w}$ in $\bar{W}$ such that $\bar{v}\bar{W} \subset \bar{W}$ and $\bar{W}\bar{w} \subset \bar{W}$. Put $W_l = \nu^{-1}(\bar{W}_l)$ and $W_r = \nu^{-1}(\bar{W}_r)$. Then if $z \,\varepsilon\, W_l$, the mapping $\bar{l}_z: \bar{w} \to \nu(z)\bar{w}$ $(\bar{w} \,\varepsilon\, \bar{W})$ is obviously holomorphic on $\bar{W}$. Hence it is clear that there exists exactly one holomorphic mapping l_z of W into itself such that $\nu \circ l_z = \bar{l}_z \circ \nu$ and $l_z(1) = z$. Similarly if $z \,\varepsilon\, W_r$, there exists just one holomorphic mapping r_z of W into itself such that $\nu(r_z(w)) = \nu(w)\nu(z)$ $(w \,\varepsilon\, W)$ and $r_z(1) = z$. For convenience we shall write $l_z w$ and $r_v w$ instead of $l_z(w)$ and $r_v(w)$ respectively $(z \,\varepsilon\, W_l, v \,\varepsilon\, W_r, w \,\varepsilon\, W)$. Let A be the analytic subgroup of G corresponding to $\mathfrak{h}_0$. Then it is obvious that $S \subset W_l$, $G \subset W_r$ and $A \subset W_l \cap W_r$. Moreover if $u, v \,\varepsilon\, W_l$, $z = l_u v$ also

lies in W_l and $l_z = l_u l_v$. This shows that the multiplication in W_l, defined by the rule $uv = l_u v$ $(u, v \varepsilon W_l)$, is associative. Similarly we define an associative multiplication in W_r by $zw = r_w z$ $(z, w \varepsilon W_r)$. It is easy to check that these two multiplications coincide on $W_r \cap W_l$.

We recall that A_c is the complex analytic subgroup of G_c corresponding to $\mathfrak{h}$. Put $\tilde{A}_c = \nu^{-1}(A_c)$. Since A contains the center[5] of G, $\tilde{A}_c = A_+ \times A$ and so it is connected. Also $A_c \subset \bar{W}_l$ and therefore $\tilde{A}_c \subset W_l$. It is easy to verify that $\tilde{A}_c$ is a group (with respect to the multiplication defined in W_l) and actually it can be regarded as a covering group of A_c under the mapping $a \rightarrow \nu(a)$ $(a \varepsilon \tilde{A}_c)$. Since $N_c^- \subset W_l$, the product na $(n \varepsilon N_c^-, a \varepsilon \tilde{A}_c)$ is well defined in W_l.

4. Holomorphic functions on W. By a holomorphic character of $\tilde{A}_c$ we mean a holomorphic function $\xi \neq 0$ on $\tilde{A}_c$ such that $\xi(ab) = \xi(a)\xi(b)$ $(a, b \varepsilon \tilde{A}_c)$. ξ being such a character, we denote by $\mathfrak{H}_\xi$ the space of all holomorphic functions f on W such that $f(l_{na}w) = f(w)\xi(a)$ $(n \varepsilon N_c^-, a \varepsilon \tilde{A}_c, w \varepsilon W)$. Our object now is to prove the following theorem.

THEOREM 1. *$\mathfrak{H}_\xi = \{0\}$ unless there exists a linear function Λ on $\mathfrak{h}$ with the following two properties:* (1) $\xi(\exp H) = e^{\Lambda(H)}$ $(H \varepsilon \mathfrak{h})$ *and* (2) $\Lambda(H_\alpha)$ *is a nonnegative integer for every positive root which is not totally positive.*[3]

For any $\bar{x} \varepsilon G_0$ and $z \varepsilon G_c$ put $z^{\bar{x}} = \bar{x}z\bar{x}^{-1}$. If y is a fixed element in G, xyx^{-1} $(x \varepsilon G)$ depends only on $\bar{x}$ and so we may denote it by $y^{\bar{x}}$. Similarly if $w = (s, x) \varepsilon S \times G = W$ and $h \varepsilon A$, $r_{h^{-1}} l_h w = (s^{\bar{h}}, x^{\bar{h}})$ and so for fixed w it depends only on $\bar{h}$. We shall denote it by $w^{\bar{h}}$. It is clear from its definition that $w \rightarrow w^{\bar{h}}$ $(w \varepsilon W)$ is a holomorphic mapping of W. A_0 being compact, we normalize its Haar measure $d\bar{h}$ in such a way that $\int_{A_0} d\bar{h} = 1$. We now need a few lemmas.

LEMMA 6. *There exists a function $\psi \varepsilon \mathfrak{H}_\xi$ such that*

$$\int_{A_0} \phi(r_x w^{\bar{h}}) d\bar{h} = \phi(x)\psi(w) \qquad (w \varepsilon W, x \varepsilon G)$$

for every $\phi \varepsilon \mathfrak{H}_\xi$. This function is unique and $\psi(1) = 1$ if $\mathfrak{H}_\xi \neq \{0\}$.

Let us first make some preliminary remarks. Every element X in $\mathfrak{g}$ defines a right-invariant holomorphic infinitesimal transformation [4] X' as

---

[5] For A contains the center of K (Weyl [9]) and K contains the center of G [7].

follows. f being a function which is defined and holomorphic around some point[6] $z \varepsilon G_c$,

$$X'f(z) = \{df(\exp(-tX)z)/dt\}_{t=0}.$$

If X, Y, Z are three elements in $\mathfrak{g}$ and $Z = [X, Y]$, it is easy to check that $Z'f = X'Y'f - Y'X'f$ in some neighborhood of z. Now since $\bar{W}$ is an open submanifold of G_c and since the mapping ν of W onto $\bar{W}$ is everywhere regular, there is exactly one holomorphic infinitesimal transformation on W ν-related to X' [4, Chap. III, § V] which we denote again by X'. Let V be an open set either in W or $\bar{W}$ and let $\mathcal{E}$ be the space of all holomorphic functions on V. Then if we associate to each $X \varepsilon \mathfrak{g}$ the linear mapping $f \to X'f$ $(f \varepsilon \mathcal{E})$ of $\mathcal{E}$ into itself, we get a representation of $\mathfrak{g}$ on $\mathcal{E}$ which can be extended (uniquely) to a representation of the universal enveloping algebra $\mathfrak{B}$ of $\mathfrak{g}$. For any $b \varepsilon \mathfrak{B}$ we denote by b' the corresponding operator on $\mathcal{E}$.

Let w be a point in G_c and f a function which is defined and holomorphic in some neighbourhood of w. Then if $X \varepsilon \mathfrak{g}$ and t is a complex variable, the function $F(t) = f(\exp(-tX)w)$ is defined and holomorphic around the origin in the complex plane and

$$\{(d^m/dt^m)F(t)\}_{t=0} = (X^m)'f(w).$$

Now let $(X_1, \cdots, X_n)$ be a base of $\mathfrak{g}$ over C. We put $X(z) = z_1X_1 + \cdots + z_nX_n$ where $z_1, \cdots, z_n$ are complex numbers. Then the function $F(z) = f((\exp - X(z))w)$ is defined and holomorphic around the origin in C^n. Let $M = (m_1, \cdots, m_n)$ be any sequence of n nonnegative integers. We write $z^M = z_1^{m_1}z_2^{m_2}\cdots z_n^{m_n}$, $M! = m_1!m_2!\cdots m_n!$, $|M| = m_1 + \cdots + m_n$ and

$$\partial^M F/\partial z^M = (\partial^{m_1+\cdots+m_n}/\partial z_1^{m_1}\cdots\partial z_n^{m_n})F.$$

Then if δ is a sufficiently small positive number, $F(z)$ is defined and holomorphic for all (z) such that $|z| = \max_i |z_i| < \delta$ and therefore

$$F(z) = \sum_M F(M)z^M/M! \qquad (|z| < \delta)$$

where $F(M) = (\partial^M F/\partial z^M)_0$, the suffix 0 denoting the value at the origin. Now replace z_i by tz_i where t is a complex number and $|t| \leqq 1$. Then

$$F(tz) = \sum_M F(M)t^{|M|}z^M/M!.$$

[6] Here $X'f(z)$ denotes the value of $X'f$ at z. A similar notation will be used in other cases as well.

On the other hand if $z=(z_1,\cdots,z_n)$ is fixed,

$$F(tz)=\sum_{m\geqq 0}(t^m/m!)(X'(z))^m f(w)$$

for $|t|$ sufficiently small. (Here $X'(z)=(X(z))'$). This follows from the fact mentioned above that

$$\{(d^m/dt^m)F(tz)\}_{t=0}=(X'(z))^m f(w).$$

Hence comparing coefficients of powers of t we get

$$(X'(z))^m f(w)=m!\sum_{m=|M|}F(M)z^M/M!.$$

Since this is true for all sufficiently small values of $|z|$, we can compare the coefficients of z^M on both sides and conclude that $F(M)=X'(M)f(w)$, where $X(M)$ is the coefficient (in $\mathfrak{B}$) of z^M in $(X(z))^m/m!$ $(m=|M|)$ and $X'(M)=(X(M))'$. This proves that

$$f((\exp-X(z))w)=\sum_M X'(M)f(w)z^M/M!$$

for all sufficiently small values of $|z|$.

Now we return to the lemma. Let ϕ and x be fixed elements in $\mathfrak{H}_\xi$ and G respectively. Put $f(w)=\phi(r_x w)$ $(w\,\varepsilon\,W)$. δ being a positive real number, let Q_δ denote the cube in C^n consisting of all points z with $|z|<\delta$. We assume that δ is so small that the following conditions are fulfilled: (1) $\exp(-X(z))\,\varepsilon\,\bar{W}$ for $z\,\varepsilon\,Q_\delta$; (2) the mapping $z\rightarrow\exp(-X(z))$ is regular and therefore open on Q_δ and hence the set $\bar{V}=\exp Q_\delta$ is an open connected neighbourhood of 1 in $\bar{W}$; (3) $\bar{V}$ is evenly covered [4, Chap. II, § VI] under the mapping ν of W onto $\bar{W}$. Let V denote the component of 1 in $\nu^{-1}(\bar{V})$. Define a function $\bar{f}$ on $\bar{V}$ by the rule $\bar{f}(\nu(w))=f(w)$ $(w\,\varepsilon\,V)$. Then f is holomorphic on $\bar{V}$. For any $\bar{h}\,\varepsilon\,A_0$ we extend $Ad(\bar{h})$ to an automorphism of $\mathfrak{B}$ and put $b^{\bar{h}}=Ad(\bar{h})b$ $(b\,\varepsilon\,\mathfrak{B})$. Let $z_i(X)$ $i=1,\cdots,n$ denote the coordinates of $X\,\varepsilon\,\mathfrak{g}$ with respect to the base $(X_1,\cdots,X_n)$. If ϵ is a positive number we denote by $\mathfrak{g}_\epsilon$ the set of all $X\,\varepsilon\,\mathfrak{g}$ such that $|z(X)|=\max_i|z_i(X)|<\epsilon$. Since A_0 is compact, we can choose ϵ so small that if $|z(X)|\leqq\epsilon$, $|z(X^{\bar{h}})|\leqq\delta/2$ for every $\bar{h}\,\varepsilon\,A_0$. Then since $\bar{f}(\exp(-X(z)))$ is holomorphic on Q_δ, it follows from our result above that

$$\bar{f}(\exp(-X))=\sum_{m\geqq 0}(X^m)'\bar{f}(1)/m! \qquad (X\,\varepsilon\,\mathfrak{g}_\delta)$$

and therefore

$$\bar{f}(\exp(-X^{\bar{h}}))=\sum_{m\geqq 0}(1/m!)((X^{\bar{h}})^m)'\bar{f}(1)/m! \qquad (X\,\varepsilon\,\mathfrak{g}_\epsilon,\bar{h}\,\varepsilon\,A_0).$$

For any $z \varepsilon Q_\delta$, let $w(z)$ denote the unique point in V such that $\nu(w(z)) = \exp(-X(z))$. Then if $z(X)$ denotes the point $(z_1(X), \cdots, z_n(X))$ in C^n, it is clear that $w(z(X^{\bar{h}})) = (w(z(X)))^{\bar{h}}$ $(X \varepsilon \mathfrak{g}_\epsilon, \bar{h} \varepsilon A_0)$ and therefore

$$f((w(z))^{\bar{h}}) = \sum_M ((X(M))^{\bar{h}})' f(1) z^M / M! \qquad (|z| < \epsilon, \bar{h} \varepsilon A_0).$$

Moreover in view of our choice of ϵ, for a fixed z the convergence is uniform with respect to $\bar{h}$. Hence

$$\int_{A_0} f((w(z))^{\bar{h}}) d\bar{h} = \sum_M (z^M/M!) \int_{A_0} ((X(M))^{\bar{h}})' f(1) d\bar{h} \qquad (|z| < \epsilon).$$

Now let $q(M) = \int_{A_0} (X(M))^{\bar{h}} d\bar{h}$. Here the integral is well defined since the elements $(X(M))^{\bar{h}}$ span a finite-dimensional subspace of $\mathfrak{B}$. Then

$$\int_{A_0} f((w(z))^{\bar{h}}) d\bar{h} = \sum_M z^M q'(M) f\ (1)/M! \qquad (|z| < \epsilon).$$

It is obvious from the definition of $q(M)$ that $[H, q(M)] = 0$ if $H \varepsilon \mathfrak{h}$. Hence by the consideration of ranks (see § 7 of [5(f)]) we see that there is exactly one element $h(M)$ in the subalgebra $\mathfrak{H}$ of $\mathfrak{B}$ generated by $(1, \mathfrak{h})$ such that $q(M) \equiv h(M) \bmod \mathfrak{B}\mathfrak{n}_-$. On the other hand $f(l_n w) = f(w)$ if $n \varepsilon N_c^-$ and therefore it follows easily that $b' f(w) = 0$ if $b \varepsilon \mathfrak{B}\mathfrak{n}_-$ and $w \varepsilon W$. Moreover since $\tilde{A}_c$ is an abelian Lie group with the Lie algebra $\mathfrak{h}$, there exists a linear function Λ on $\mathfrak{h}$ such that $\xi(\exp H) = e^{\Lambda(H)}$ $(H \varepsilon \mathfrak{h})$. Then $f(l_a w) = e^{\Lambda(H)} f(w)$ if $a = \exp H \varepsilon \tilde{A}_c$. We extend the mapping $H \to -\Lambda(H)$ $(H \varepsilon \mathfrak{h})$ to a (uniquely determined) homomorphism μ_Λ of $\mathfrak{H}$ into C such that $\mu_\Lambda(1) = 1$. Then it is clear that $h' f(w) = \mu_\Lambda(h) f(w)$ for $h \varepsilon \mathfrak{H}$ and $w \varepsilon W$. Hence

$$q'(M) f(1) = h'(M) f(1) = \mu_\Lambda(h(M)) f(1),$$

and therefore

$$\int_{A_0} f((w(z))^{\bar{h}}) d\bar{h} = \sum_M z^M \mu_\Lambda(h(M)) f(1)/m! \qquad (|z| < \epsilon).$$

Now if we put $\Phi(x, w) = \int_{A_0} \phi(r_x w^{\bar{h}}) d\bar{h}$ and recall that $f(1) = \phi(x)$ we get

$$\Phi(x, w(z)) = \phi(x) \sum_M z^M \mu_\Lambda(h(M))/M!$$

provided $|z| < \epsilon$. It is clear that the set of all points $w(z)$ $(|z| < \epsilon)$ is a neighbourhood of 1 in W. Hence from the principle of analytic continuation, there exists at most one holomorphic function ψ on W such that

$$\psi(w(z)) = \sum_M z^M \mu_\Lambda(h(M))/M!$$

if $|z| < \epsilon$. On the other hand if $\mathfrak{H}_\xi \neq \{0\}$ we can choose a function $\phi_0 \neq 0$ in $\mathfrak{H}_\xi$. Since every element in W is of the form $l_s x$ $(s \varepsilon S, x \varepsilon G)$, it is obvious that $\phi_0(x_0) \neq 0$ for some $x_0 \varepsilon G$. Moreover since A_0 is compact, the function

$$\Phi_0(x_0, w) = \int_{A_0} \phi_0(r_{x_0} w^{\bar{h}}) d\bar{h} \qquad (w \varepsilon W)$$

is obviously holomorphic on W and therefore the same holds for $\Phi_0(x_0, w)/\phi_0(x_0)$. But if we apply the above relation to ϕ_0, we get

$$\Phi_0(x_0, w(z))/\phi_0(x_0) = \sum_M z^M \mu_\Lambda(h(M))/M! \qquad (|z| < \epsilon).$$

This shows that the function ψ certainly exists (if $\mathfrak{H}_\xi \neq \{0\}$). On the other hand it is obvious that $\Phi_0(x_0, l_{na} w) = \xi(a)\Phi_0(x_0, w)$ $(n \varepsilon N_c^-, a \varepsilon \tilde{A}_c, w \varepsilon W)$ and therefore $\psi \varepsilon \mathfrak{H}_\xi$. Furthermore since $\Phi(x, w)$ and $\phi(x)\psi(w)$ coincide on a neighbourhood of 1 in W and since they are both holomorphic in w, they must coincide everywhere. This proves that

$$\int_{A_0} \phi(r_x w^{\bar{h}}) d\bar{h} = \phi(x)\psi(w) \qquad (x \varepsilon G, w \varepsilon W),$$

and the uniqueness of ψ is obvious from this formula. In particular if we put $\phi = \phi_0$, $x = x_0$ and $w = 1$, we get $\psi(1) = 1$. Finally if $\mathfrak{H}_\xi = \{0\}$, ψ must also be zero and so ψ is unique in any case. Thus the lemma is proved.

Let Λ denote, as above, the linear function on $\mathfrak{h}$ such that $\xi(\exp H) = e^{\Lambda(H)}$ $(H \varepsilon \mathfrak{h})$.

LEMMA 7. *Let $\phi \neq 0$ be a function in $\mathfrak{H}_\xi$. Suppose there exists a linear function Λ' on $\mathfrak{h}$ such* that

$$\phi(r_a w) = e^{\Lambda'(H)} \phi(w) \qquad (w \varepsilon W, H \varepsilon \mathfrak{h}_0)$$

where $a = \exp H \varepsilon A$. Then $\Lambda - \Lambda'$ is a linear combination of positive roots with coefficients which are nonnegative integers.

Let V be an open connected neighbourhood of 1 in W such that V is mapped in a one-one fashion on $\bar{V} = \nu(V)$ under the mapping ν. Define a function $\bar{\phi}$ on $\bar{V}$ by setting $\bar{\phi}(\nu(w)) = \phi(w)$ $(w \varepsilon V)$. Let $\alpha_1, \cdots, \alpha_r$ be all the distinct positive roots of $\mathfrak{g}$. We put $X_i = X_{\alpha_i}$ $1 \leqq i \leqq r$. Then $(X_1, \cdots, X_r)$ is a base for $\mathfrak{n}_+$ over C. Put $X(z) = z_1 X_1 + \cdots + z_r X_r$ (where the z_i's are complex numbers) and for any positive ϵ let $\mathfrak{n}_+(\epsilon)$ denote the

neighbourhood of zero in $\mathfrak{n}_+$ consisting of all $X(z)$ with $|z| = \max_i |z_i| < \epsilon$. We can choose ϵ so small that $\exp(-X(z)) \varepsilon \bar{V}$ and

$$\bar{\phi}(\exp(-X(z))) = \sum_M X'(M)\bar{\phi}(1)z^M/M!$$

if $|z| < \epsilon$. (Here the notation is similar to what we used in the proof of Lemma 6. M is a sequence $(m_1, \cdots, m_r)$ of r nonnegative integers and $X(M)$ is the coefficient (in $\mathfrak{B}$) of z^M in $(1/m!)(X(z))^m$ where $m = m_1 + \cdots + m_r$. Moreover $M! = m_1! \cdots m_r!$). Choose a positive real δ such that $(X(z))^{\bar{h}} \varepsilon \mathfrak{n}_+(\epsilon)$ for all $\bar{h} \varepsilon A_0$ if $|z| < \delta$. Then it is obvious that

$$\bar{\phi}(\exp(-X(z))^{\bar{h}}) = \sum_M ((X(M))^{\bar{h}})'\bar{\phi}(1)z^M/M! \qquad (|z| < \delta, \bar{h} \varepsilon A_0).$$

However if $h = \exp H$ $(H \varepsilon \mathfrak{h}_0)$, it is obvious that

$$\phi(w^{\bar{h}}) = \phi(l_h r_{h^{-1}} w) = \exp(\Lambda(H) - \Lambda'(H))\phi(w) \qquad (w \varepsilon W).$$

Hence if $|z| < \delta$,

$$\bar{\phi}(\exp(-X(z))^{\bar{h}}) = \exp(\Lambda(H) - \Lambda'(H))\bar{\phi}(\exp(-X(z))).$$

On the other hand if $M = (m_1, \cdots, m_r)$, it is clear that

$$(X(M))^{\bar{h}} = e^{\alpha_M(H)}X(M),$$

where $\alpha_M = m_1\alpha_1 + \cdots + m_r\alpha_r$. Therefore

$$\exp(\Lambda(H) - \Lambda'(H)) \sum_M X'(M)\bar{\phi}(1)z^M/M! \\ = \sum_M e^{\alpha_M(H)}X'(M)\bar{\phi}(1)z^M/M!$$

for all $H \varepsilon \mathfrak{h}_0$ and all $z = (z_1, \cdots, z_r)$ with $|z| < \delta$. Hence comparing coefficients of z^M we get $X'(M)\bar{\phi}(1) = 0$ unless $\Lambda - \Lambda' = \alpha_M$. On the other hand $X'(M)\bar{\phi}(1)$ cannot be zero for all M. For otherwise $\bar{\phi}(\exp(-X(z)) = 0$ if $|z| < \delta$. But since $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$, the elements of the form $na\exp(-X(z))$ $(a \varepsilon A_c, n \varepsilon N_c^-, |z| < \delta)$ cover a neighbourhood of 1 in $\bar{W}$. Since $\phi(l_{na}w) = \xi(a)\phi(w)$, it is obvious that

$$\bar{\phi}(na\exp(-X(z))) = \xi(a)\bar{\phi}(\exp(-X(z)) = 0$$

if $na\exp(-X(z)) \varepsilon \bar{V}$ and $|z| < \delta$. This shows that $\bar{\phi}$ vanishes identically on a neighbourhood of 1 in $\bar{V}$ and therefore ϕ is also zero on some neighbourhood of 1 in W. But then ϕ, being holomorphic, must be zero everywhere on W. Since this contradicts our hypothesis, $\Lambda - \Lambda' = \alpha_M$ for some M and so the lemma is proved.

Every $X \varepsilon \mathfrak{g}$ may be regarded as a (left-invariant) holomorphic infini-

tesimal transformation on G_c [4, Chap. IV] and therefore also on its open submanifold $\bar{W}$. Then there is exactly one holomorphic infinitesimal transformation on W which is ν-related to X. We denote it also by X. Let V be an open set either in W or $\bar{W}$ and let $\mathcal{E}$ be the space of all holomorphic functions on V. Then these operations of $\mathfrak{g}$ define a representation of $\mathfrak{g}$ on $\mathcal{E}$ which may be extended uniquely to a representation of $\mathfrak{B}$. If $b \varepsilon \mathfrak{B}$ and $f \varepsilon \mathcal{E}$, we denote by $bf(w)$ the value of bf at w $(w \varepsilon V)$. Let $\mathfrak{X}$ be the subalgebra of $\mathfrak{B}$ generated by $(1, \mathfrak{k})$.

LEMMA 8. *Let ψ be the function of Lemma 6. Then $H\psi = \Lambda(H)\psi$ for $H \varepsilon \mathfrak{h}$ and $X_\alpha \psi = 0$ for every positive root α. Moreover the functions $b\psi$ $(b \varepsilon \mathfrak{X})$ span a finite-dimensional subspace of $\mathfrak{H}_\xi$.*

Since r_u and l_v $(u \varepsilon W_r, v \varepsilon W_l)$ commute, it is an easy matter to verify that if $f \varepsilon \mathfrak{H}_\xi$ and $X \varepsilon \mathfrak{g}$ then Xf is also in $\mathfrak{H}_\xi$. Therefore we get a representation of $\mathfrak{B}$ on $\mathfrak{H}_\xi$. Now for any $\phi \varepsilon \mathfrak{H}_\xi$, consider the function

$$\Phi(x, w) = \phi(r_x w) \qquad (x \varepsilon G, w \varepsilon W)$$

on $G \times W$. Since $(x, w) \rightarrow \nu(r_x w) = \nu(w)\bar{x}$ is a (real) analytic mapping of $G \times W$ into $\bar{W}$, it is clear that $(x, w) \rightarrow r_x w$ is also an analytic mapping of $G \times W$ into W. If $W \varepsilon \mathfrak{g}_0$ it is obvious from the definition of $X\phi$ that

$$X\phi(w) = \{d\Phi(\exp tX, w)/dt\}_{t=0} \qquad (t \varepsilon R).$$

Moreover if $X, Y \varepsilon \mathfrak{g}_0$ and $Z = X + (-1)^{\frac{1}{2}} Y \varepsilon \mathfrak{g}$, $Z\phi = X\phi + (-1)^{\frac{1}{2}}(Y\phi)$. Therefore if Λ' is a linear function on $\mathfrak{h}$, it follows from the above differential equation that $H\phi = \Lambda'(H)\phi$ for every $H \varepsilon \mathfrak{h}$ if and only if

$$\Phi(\exp H, w) = e^{\Lambda'(H)}\phi(w)$$

for all $H \varepsilon \mathfrak{h}_0$ and $w \varepsilon W$. In particular if we apply this criterion to ψ and take into account the fact (which follows from Lemma 6) that $\psi(w^{\bar{h}}) = \psi(w)$ $(\bar{h} \varepsilon A_0, w \varepsilon W)$, we get $H\psi = \Lambda(H)\psi$ $(H \varepsilon \mathfrak{h})$. But then if $\phi = X_\alpha \psi$ it is clear that $H\phi = (\Lambda(H) + \alpha(H))\phi$ $(H \varepsilon \mathfrak{h})$ and therefore by the above criterion

$$\phi(r_h w) = \exp(\Lambda(H) + \alpha(H))\phi(w) \qquad (H \varepsilon \mathfrak{h}_0, w \varepsilon W)$$

where $h = \exp H \varepsilon A$. α being a positive root, we can conclude from Lemma 7 that $\phi = X_\alpha \psi = 0$.

To prove the last assertion we may assume that $\mathfrak{H}_\xi \neq \{0\}$. Let K and K' be the analytic subgroups of G corresponding to $\mathfrak{k}_0$ and $\mathfrak{k}_0' = [\mathfrak{k}_0, \mathfrak{k}_0]$ respectively. Then K' is compact and semisimple[4] and the function

$$\Psi(u, w) = \psi(r_u w) \qquad (u \varepsilon K', w \varepsilon W)$$

is continuous on $K' \times W$. Moreover $\Psi(1,1) = \psi(1) = 1$. Therefore from the Peter-Weyl Theorem for K', there exists an irreducible character χ of K' with the property that the function

$$\Psi'(u,w) = \int_{K'} \chi(v^{-1}) \Psi(uv,w)\, dv$$

is not identically zero on $K' \times W$. (Here dv is the Haar measure on K'). Since $\Psi(uv,w) = \Psi(v, r_u w)$, it follows that $\Psi'(u,w) = \Psi'(1, r_u w)$. Therefore if $\psi'(w) = \Psi'(1,w)$ $(w \,\varepsilon\, W)$, it is obvious that $\psi' \,\varepsilon\, \mathfrak{H}_\xi$ and it is not zero. Now if $h \,\varepsilon\, A$,

$$\psi'(w^{\bar{h}}) = \int_{K'} \chi(v^{-1})\, \psi(r_v w^{\bar{h}})\, dv.$$

Moreover K' is a normal subgroup of K and it is clear that $\chi(h^{-1}v^{-1}h) = \chi(v^{-1})$. Therefore

$$\int_{K'} \chi(v^{-1}) \psi(r_v w^{\bar{h}})\, dv = \int_{K'} \chi(v^{-1}) \psi((r_v w)^{\bar{h}})\, dv = \int_{K'} \chi(v^{-1}) \psi(r_v w)\, dv = \psi'(w),$$

since $\psi(z^{\bar{h}}) = \psi(z)$ $(z \,\varepsilon\, W)$. This shows that $\psi'(w^{\bar{h}}) = \psi'(w)$ and therefore from Lemma 6,

$$\psi'(w) = \int_{A_0} \psi'(w^{\bar{h}})\, d\bar{h} = \psi'(1)\psi(w) \qquad (w \,\varepsilon\, W).$$

This proves that the function ψ and ψ' differ only by a constant factor which however cannot be zero since neither ψ nor ψ' is zero. For any fixed $u \,\varepsilon\, K'$ put $\psi_u'(w) = \Psi'(u,w) = \psi'(r_u w)$. Then it is clear from the definition of $\Psi'(u,w)$ that $\psi_u' \,\varepsilon\, \mathfrak{H}_\xi$ and the dimension of the subspace V of $\mathfrak{H}_\xi$ spanned by all ψ_u' $(u \,\varepsilon\, K')$ is finite. For any $\phi \,\varepsilon\, V$ define $\phi_u(w) = \phi(r_u w)$ $(u \,\varepsilon\, K', w \,\varepsilon\, W)$. Then if to each $u \,\varepsilon\, K'$ we associate the linear mapping $\sigma(u): \phi \rightarrow \phi_u$ of V into itself, we get a representation σ of K' on the finite-dimensional space V. We denote the corresponding representation of $\mathfrak{k}_0'$ also by σ. Since $\sigma(u)\phi(w) = \phi(r_u w)$ it follows immediately that $\sigma(X)\phi = X\phi$ $(X \,\varepsilon\, \mathfrak{k}_0', \phi \,\varepsilon\, V)$. On the other hand if $\mathfrak{c}_0$ is the center of $\mathfrak{k}_0$ and $h = \exp H \,\varepsilon\, A$ $(H \,\varepsilon\, \mathfrak{c}_0)$ it is clear that $\psi_u(r_h w) = \psi(r_h r_u w) = e^{\Lambda(H)}\psi(r_u w) = e^{\Lambda(H)}\psi_u(w)$. Since V is spanned by the functions ψ_u, we can conclude that $\phi(r_h w) = e^{\Lambda(H)}\phi(w)$ for all $\phi \,\varepsilon\, V$. But as we have seen earlier this implies that $H\phi = \Lambda(H)\phi$ and therefore V is invariant under the operations of $\mathfrak{k}_0' + \mathfrak{c}_0 = \mathfrak{k}_0$ and so also under those of $\mathfrak{k}$. Since $\psi \,\varepsilon\, V$, the last assertion of the Lemma follows immediately.

It is now obvious that Theorem 1 is a direct consequence of Lemma 8 and Theorem 1 of [5(f)].

5. Representations on a Hilbert space of holomorphic functions. We shall now prove a converse of Theorem 1. Since $\tilde{A}_c$ is the direct product of A_+ and A and A_+ is simply connected, $\exp H = 1$ in $\tilde{A}_c$ $(H \varepsilon \mathfrak{h})$ if and only if $H \varepsilon \mathfrak{h}_0$ and $\exp H = 1$ in A. On the other hand if D and A' are the analytic subgroups of A corresponding to[3] $\mathfrak{c}_0$ and $\mathfrak{h}_0' = \mathfrak{h}_0 \cap \mathfrak{k}_0'$, A is the direct product of D and A' and D is simply connected.[4] Therefore $\exp H \neq 1$ in A unless $H \varepsilon \mathfrak{h}_0'$. Now suppose $H \varepsilon \mathfrak{h}_0'$. Since $A' \subset K'$ and K' is compact, $\exp H = 1$ if and only if it lies in the kernel of every finite-dimensional irreducible representation of K'. But K' is simply connected[4] and therefore there is a one-one correspondence between finite-dimensional representations of K' and those of $\mathfrak{k}'$. So it is clear that $\exp H = 1$ if and only if $e^{\Lambda(H)} = 1$ for all weights Λ (with respect to $\mathfrak{h}'$) of all finite-dimensional representations of $\mathfrak{k}'$. Since we may identify compact roots of $\mathfrak{g}$ with roots of $\mathfrak{k}'$, it follows[7] that a linear function Λ on $\mathfrak{h}$ coincides on $\mathfrak{h}'$ with the weight of some finite-dimensional representation of $\mathfrak{k}'$, if and only if $\Lambda(H_\alpha)$ is an integer for every compact root α. Therefore *in order that there should exist a (holomorphic) character ξ of $\tilde{A}_c$ satisfying the equation $\xi(\exp H) = e^{\Lambda(H)}$ $(H \varepsilon \mathfrak{h})$ it is necessary and sufficient that $\Lambda(H_\alpha)$ should be an integer for every compact root α.*

Now let μ and ω be two nonnegative (Haar) measurable functions on G_0. We assume that μ is not identically zero, ω is bounded on every compact set and $\mu(\bar{x}\bar{y}) \leqq \mu(\bar{x})\omega(\bar{y})$ $(\bar{x}, \bar{y} \varepsilon G_0)$. Then $\mu(\bar{y}) \leqq \mu(\bar{x})\omega(\bar{x}^{-1}\bar{y})$ and since μ is not identically zero it follows that $\mu(\bar{x}) \neq 0$. Moreover

$$\{\mu(\bar{x}\bar{y})\}^{-1} \leqq \{\mu(\bar{x})\}^{-1}\omega(\bar{y}^{-1})$$

and since $\omega(\bar{y})$ (and therefore also $\omega(\bar{y}^{-1})$) is bounded on every compact set, it follows from the above inequalities that the same holds for both μ and $1/\mu$. Hence if $d\bar{x}$ is the Haar measure on G_0, the two measures $\mu(\bar{x})d\bar{x}$ and $d\bar{x}$ are absolutely continuous with respect to each other.

Let Λ be a *real*[2] linear function on $\mathfrak{h}$. We assume that $\Lambda(H_\alpha)$ is a nonnegative integer for every positive root α which is not totally positive. Then, as we have seen above, there exists a holomorphic character ξ of $\tilde{A}_c$ such that $\xi(\exp H) = e^{\Lambda(H)}$ $(H \varepsilon \mathfrak{h})$. Let f be a complex-valued function on W such that $f(l_a w) = \xi(a)f(w)$ for all $a \varepsilon \tilde{A}_c$ and $w \varepsilon W$. If Z is the center of G, $Z \subset A$[5] and therefore $f(l_z w) = \xi(z)f(w)$ $(z \varepsilon Z)$ and $|\xi(z)| = 1$ because Λ is real. This shows that $|f(zx)| = |f(x)|$ $(z \varepsilon Z, x \varepsilon G)$ and so $|f(x)|$ depends only on $\bar{x}$. Hence if $|f(x)|$ happens to be a measurable function of $\bar{x}$,

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[7] This can be deduced easily from Theorem 1 of [5(a)]. See also Weyl [9].

we can consider the integral $\int_{G_0} |f(x)|^2\mu(\bar{x})d\bar{x}$. Now let $\mathfrak{H}_\Lambda(\mu)$ denote the space of all holomorphic functions f on W satisfying the following two conditions:

(1) $f(l_{na}w) = \xi(a)f(w) \qquad (n \in N_c^-, a \in \tilde{A}_c, w \in W)$

(2) $\|f\|^2 = \int_{G_0} |f(x)|^2\mu(\bar{x})d\bar{x} < \infty.$

We put $\mathfrak{H} = \mathfrak{H}_\Lambda(\mu)$ for convenience. Then every $f \in \mathfrak{H}$ is certainly continuous on G. Therefore since the measures $d\bar{x}$ and $\mu(\bar{x})d\bar{x}$ are absolutely continuous with respect to each other, $\|f\|$ is positive unless f vanishes identically on G. But then in view of condition (1) above, $f = 0$. Therefore in order to prove that $\mathfrak{H}$ is a Hilbert space, it only remains to prove that it is complete. After this has been done we intend to define a representation of G on $\mathfrak{H}$ and prove that $\mathfrak{H} = \mathfrak{H}_\Lambda(\mu) \neq \{0\}$ for a suitable choice of μ.

Let x_0 be a fixed element in G and let $X_1, \cdots, X_n$ be a base for $\mathfrak{g}$ over C. If $X \in \mathfrak{g}$, we denote by $z_1(X), \cdots, z_n(X)$ the coordinates of X with respect to this base and for any positive number ϵ we define (as before) $\mathfrak{g}_\epsilon$ to be the set of all $X \in \mathfrak{g}$ such that $|z(X)| = \max_i |z_i(X)| < \epsilon$. Now choose ϵ so small that the following conditions hold: (1) the mapping $X \to \exp(-X)$ of $\mathfrak{g}$ into G_c is univalent and regular on $\mathfrak{g}_\epsilon$, (2) $\exp(-X)\bar{x}_0 \in \bar{W}$ for $X \in \mathfrak{g}_\epsilon$, (3) the set $\bar{V} = \exp(\mathfrak{g}_\epsilon)\bar{x}_0$ is evenly covered under the mapping ν of W onto $\bar{W}$. Let V denote the connected component of x_0 in $\nu^{-1}(\bar{V})$. Then V and $\bar{V}$ are both open and ν defines a one-one regular holomorphic mapping of V onto $\bar{V}$.

For any $X \in \mathfrak{g}$, consider the endomorphism

$$(1 - \exp(-adX))/adX = \sum_{m \geqq 0} (-1)^m (adX)^m/(m+1)!$$

of the complex vector space $\mathfrak{g}$. We denote by $\Delta(X)$ its determinant. Then $\Delta(X)$ is clearly a holomorphic function on $\mathfrak{g}$ and a well-known computation [4, p. 157] shows that if $x^* = \exp(-X)\bar{x}_0$,

$$dx^* = d(x^*)^{-1} = |\Delta(X)|^2 |dX|^2 \qquad (X \in \mathfrak{g}_\epsilon)$$

where dx^* is the Haar measure on G_c and $|dX|^2$ is the Euclidean measure on $\mathfrak{g}$ (regarded as a vector space over R). Since $\Delta(0) = 1$, we may assume that ϵ is so small that $|\Delta(X)| \geqq \frac{1}{2}$ on $\mathfrak{g}_\epsilon$.

On the other hand we know (see Corollary to Lemma 1 and Lemma 26 of [5(b)]) that $(n, a, \bar{x}) \to na\bar{x}$ $(n \in N_c^-, a \in A_+, \bar{x} \in G_0)$ is a one-one regular mapping of $N_c^- \times A_+ \times G_0$ onto $\bar{W}$ and an easy computation shows that

$$dx^* = e^{4\rho(\log a)} dn da d\bar{x},$$

where $x^* = na\bar{x}$ and dx^*, dn, da are the (suitably normalised) Haar measures of G_c, N_c^- and A_+ respectively. Let V' be the set of all points $(n, a, \bar{x}) \varepsilon N_c^- \times A_+ \times G_0$ such that $na\bar{x} \varepsilon \bar{V}$ and let V_1, V_2, V_3 denote the projections of V' on each of the factors N_c^-, A_+, G_0 respectively. Then they are all open and since the closure of $\mathfrak{g}_\epsilon$ is compact, it is clear that the closures of $\bar{V}, V', V_1, V_2, V_3$ are also all compact. We can choose open neighbourhoods V_1', V_2', V_3' of $1, 1, \bar{x}_0$ in N_c^-, A_+, G_0 respectively such that $V_1'V_2'V_3' \subset \bar{V}$. Also choose a positive $\delta < \epsilon$ such that $\exp(-X)\bar{x}_0 \varepsilon V_1'V_2'V_3'$ if $X \varepsilon \mathfrak{g}_\delta$. If f is any function in $\mathfrak{H}$, we denote by $\bar{f}$ the function on $\bar{V}$ given by $\bar{f}(\nu(w)) = f(w)$ $(w \varepsilon V)$. Then

$$\int_{\mathfrak{g}_\delta} |\bar{f}(\exp(-X)\bar{x}_0)|^2 \, |dX|^2 \leqq 4 \int_{\mathfrak{g}_\delta} |\bar{f}(\exp(-X)x_0)|^2 \, |\Delta(X)|^2 \, |dX|^2$$

$$\leqq 4 \int_{V_1 \times V_2 \times V_3} |\bar{f}(na\bar{x})|^2 e^{4\rho(\log a)} dn da d\bar{x} = M_1 \int_{V_3} |\bar{f}(\bar{x})|^2 d\bar{x}$$

where $M_1 = 4 \int_{V_1} \exp(2\Lambda(\log a) + 4\rho(\log a)) da \int_{V_2} dn$. On the other hand since the closure of V_3 is compact, $1/\mu$ is bounded on V_3. Hence

$$\int_{V_3} |\bar{f}(\bar{x})|^2 d\bar{x} \leqq M_2 \int_{V_3} |\bar{f}(\bar{x})|^2 \mu(\bar{x}) d\bar{x} \leqq M_2 \, \| f \|^2$$

where M_2 is an upper bound for μ^{-1} on V_3. This proves that

$$\int_{\mathfrak{g}_\delta} |\bar{f}(\exp(-X)\bar{x}_0)|^2 \, |dX|^2 \leqq M \, \| f \|^2$$

where $M = M_1 M_2$. Since $\bar{f}(\exp(-X)\bar{x}_0)$ is a holomorphic function of X on $\mathfrak{g}_\epsilon$, it follows from a classical argument (see Bochner and Martin [2, p. 117]) that if $\| f \|$ tends to zero, $\bar{f}(\exp(-X)\bar{x}_0)$ tends to zero uniformly on every compact subset of $\mathfrak{g}_\delta$. This proves that $\| f \| \to 0$ implies the uniform convergence of f to zero on some neighbourhood of x_0 in W. Moreover since x_0 can be any point in G and since $f(nax) = \xi(a) f(x)$ $(n \varepsilon N_c^-, a \varepsilon A_+, x \varepsilon G)$ it is clear that the same conclusion holds in some neighbourhood of any given point in W. Therefore if f_m is a Cauchy sequence in $\mathfrak{H}$, it converges uniformly on every compact set in W and hence the limit function f is holomorphic on W and clearly $f(l_{na}w) = \xi(a) f(w)$. Moreover it then follows by well-known elementary arguments that

$$\| f \|^2 = \int_{G_0} |f(x)|^2 \mu(\bar{x}) d\bar{x} = \operatorname*{Lim}_{m \to \infty} \| f_m \|^2 < \infty$$

and $\| f - f_m \| \to 0$. This proves that f lies in $\mathfrak{H}$ and f_m converges to f in $\mathfrak{H}$. Therefore $\mathfrak{H}$ is complete.

Now we define a representation π of G on $\mathfrak{H}$ as follows. If $f \varepsilon \mathfrak{H}$ and $y \varepsilon G$, $\pi(y)f$ is the function whose value at w is $f(r_y w)$ $(w \varepsilon W)$. Since $\mu(\bar{x}\bar{y}^{-1}) \leqq \mu(\bar{x})\omega(\bar{y}^{-1})$,

$$\int_{G_0} |f(xy)|^2 \mu(\bar{x})\, d\bar{x} \leqq \omega(\bar{y}^{-1}) \| f \|^2,$$

and so it follows easily that $\pi(y)f \varepsilon \mathfrak{H}$ and $\| \pi(y)f \|^2 \leqq \omega(\bar{y}^{-1}) \| f \|^2$. This shows moreover that the operators $\pi(y)$ remain uniformly bounded on any compact set. Now let $V = V^{-1}$ be any compact neighbourhood of 1 in G and let $\bar{V}$ be its image in G_0 under the mapping $x \to \bar{x}$. For any given $\epsilon > 0$, we can choose a compact set G_1 in G_0 such that

$$\int_{^cG_1} |f(x)|^2 \mu(\bar{x})\, d\bar{x} \leqq \epsilon^2.$$

($^cG_1$ is the complement of G_1 in G_0). Put $G_2 = G_1\bar{V}$. Then G_2 is also compact and if $y \varepsilon V$, $(^cG_2)\bar{y} \subset {}^cG_1$. Hence

$$\int_{^cG_2} |f(xy)|^2 \mu(\bar{x})\, d\bar{x} \leqq \omega(\bar{y}^{-1}) \int_{(^cG_1)} |f(x)|^2 \mu(\bar{x})\, d\bar{x} \leqq M^2\epsilon^2,$$

where M^2 is an upper bound for $\omega(\bar{y})$ on $\bar{V}$. Moreover

$$\| \pi(y)f - f \|^2 = \int_{G_2} |f(xy) - f(x)|^2 \mu(\bar{x})\, d\bar{x} + \int_{^cG_2} |f(xy) - f(x)|^2 \mu(\bar{x})\, d\bar{x}.$$

But

$$\int_{^cG_2} |f(xy) - f(x)|^2 \mu(\bar{x})\, d\bar{x}$$

$$\leqq [(\int_{^cG_2} |f(xy)|^2 \mu(\bar{x})\, d\bar{x})^{\frac{1}{2}} + (\int_{^cG_2} |f(x)|^2 \mu(\bar{x})\, d\bar{x})^{\frac{1}{2}}]^2 \leqq \{(M+1)\epsilon\}^2,$$

and since G_2 is compact,

$$\operatorname*{Lim}_{y \to 1} \int_{G_2} |f(xy) - f(x)|^2 \mu(\bar{x})\, d\bar{x} = 0.$$

As ϵ is arbitrary, this shows that $\operatorname*{Lim}_{y \to 1} \| \pi(y)f - f \| = 0$. Moreover it is obvious that $\pi(xy) = \pi(x)\pi(y)$ $(x, y \varepsilon G)$ and therefore π is a representation [5(b), p. 201] of G on $\mathfrak{H}$.

It is obvious that $\mathfrak{H}$ is contained in the space $\mathfrak{H}_\xi$ of Lemma 6.

LEMMA 9. *If $\mathfrak{H} = \mathfrak{H}_\Lambda(\mu) \neq \{0\}$, the function ψ of Lemma 6 lies in $\mathfrak{H}$.*

For if we choose $\phi_0 \neq 0$ in $\mathfrak{H}$, it is clear that $\phi_0(x_0) \neq 0$ for some $x_0 \varepsilon G_0$. Then it follows from Lemma 6 that

$$\int_{A_0} \phi_0(y^{\bar{h}} x_0)\, d\bar{h} = \phi_0(x_0)\psi(y),$$

and therefore

$$|\phi_0(x_0)|^2 \int_{G_0} |\psi(y)|^2 \mu(\bar{y})d\bar{y} \leqq \int_{G_0} \mu(\bar{y})d\bar{y} \int_{A_0} |\phi_0(y^{\bar{h}}x_0)|^2 d\bar{h}$$

$$= \int_{A_0} d\bar{h} \int_{G_0} |\phi_0(y^{\bar{h}}x_0)|^2 \mu(\bar{y})d\bar{y}.$$

But if $h \varepsilon A$,

$$\int_{G_0} |\phi_0(y^h x_0)|^2 \mu(\bar{y})d\bar{y} = \int_{G_0} |\phi_0(hyh^{-1}x_0)|^2 \mu(\bar{y})d\bar{y}$$

$$= \int_{G_0} |\phi_0(y)|^2 \mu(\bar{y}\bar{x}_0^{-1}\bar{h})d\bar{y} \leqq \omega(\bar{x}_0^{-1}\bar{h}) \| \phi_0 \|^2,$$

since $|\xi(h)| = 1$. Therefore if M is an upper bound for ω on the compact set $\bar{x}_0^{-1}A_0$, it follows that $|\phi_0(x_0)|^2 \| \psi \|^2 \leqq M \| \phi_0 \|^2$. Since $\phi_0(x_0) \neq 0$, we conclude that $\| \psi \|^2 < \infty$ and therefore $\psi \varepsilon \mathfrak{H}$.

LEMMA 10. *Let ϕ be any element in $\mathfrak{H}$ which is well-behaved*[8] *under π. Then $\pi(X)\phi = X\phi$ $(X \varepsilon \mathfrak{g})$. Moreover if $\mathfrak{H} \neq \{0\}$, ψ is well-behaved under π.*

We have seen that if ϕ tends to ϕ_0 in $\mathfrak{H}$ then $\phi(w)$ tends to $\phi_0(w)$ uniformly on every compact set in W. In particular if ϕ is well-behaved under π and $X \varepsilon \mathfrak{g}_0$, $(1/t)\{\pi(\exp tX)\phi - \phi\}$ $(t \varepsilon R)$ tends to $\pi(X)\phi$ in $\mathfrak{H}$ as $t \to 0$. On the other hand it is obvious that $(1/t)\{\pi(\exp tX)\phi - \phi\}$ tends to $X\phi$ uniformly on every compact set in W. Therefore $X\phi = \pi(X)\phi$ and by linearity this remains true if $X \varepsilon \mathfrak{g}$. Also we know that $\pi(z) = \xi(z)\pi(1)$ $(z \varepsilon Z)$. Therefore it follows from Theorem 4 of [5(b), p. 224] that the space of well-behaved elements is dense in $\mathfrak{H}$. On the other hand if $h \varepsilon A$, it is obvious that $\xi(h^{-1})\pi(h)$ depends only on $\bar{h}$ and so we may denote it by $\bar{\pi}(\bar{h})$. Then $\bar{h} \to \bar{\pi}(\bar{h})$ $(\bar{h} \varepsilon A_0)$ is a representation of A_0 on $\mathfrak{H}$ and

$$E = \int_{A_0} \bar{\pi}(\bar{h})d\bar{h}$$

is a bounded operator with $E^2 = E$. Moreover since A_0 is compact, it follows easily (see Lemma 29 of [5(b)]) that if ϕ is well-behaved under π, the same is true of $E\phi$. Moreover if we regard

$$E\phi = \int_{A_0} \bar{\pi}(\bar{h})\phi d\bar{h} \qquad (\phi \varepsilon \mathfrak{H})$$

as the limit of a sum in $\mathfrak{H}$, it is clear from the remarks on convergence made above that[6]

$$E\phi(w) = \int_{A_0} \bar{\pi}(\bar{h})\phi(w)d\bar{h} \qquad (w \varepsilon W).$$

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[8] We use here and in the rest of this paper the terminology of [5(b)].

But if $h \varepsilon A$,

$$\bar{\pi}(\bar{h})\phi(w) = \xi(h^{-1})\pi(h)\phi(w) = \xi(h^{-1})\phi(r_h w) = \xi(h^{-1})\phi(l_h w^{\bar{a}}) = \phi(w^{\bar{a}})$$

where $a = h^{-1}$. Therefore, from Lemma 6,

$$E\phi(w) = \phi(1)\psi(w) \quad \text{or} \quad E\phi = \phi(1)\psi.$$

Now assume that $\mathfrak{H} \neq \{0\}$. Then from Lemma 9, $\psi \varepsilon \mathfrak{H}$ and so we can choose a sequence ϕ_m of well-behaved elements in $\mathfrak{H}$ such that $\phi_m \to \psi$ in $\mathfrak{H}$. Since E is bounded

$$\phi_m(1)\psi = E\phi_m \to E\psi = \psi(1)\psi = \psi$$

from Lemma 6. Therefore $\phi_m(1) \neq 0$ if m is sufficiently large and since $E\phi_m$ is well-behaved the same holds for $\psi = \{\phi_m(1)\}^{-1}E\phi_m$.

LEMMA 11. *Suppose $\mathfrak{H} \neq \{0\}$ and $\mathfrak{H}_1$ is the smallest closed subspace of $\mathfrak{H}$ containing ψ which is invariant under $\pi(G)$. Then the representation of G defined on $\mathfrak{H}_1$ under π is irreducible and quasi-simple.*[8]

Let $\mathfrak{H}_2 \neq \{0\}$ be any closed invariant subspace of $\mathfrak{H}_1$. Choose $\phi \neq 0$ in $\mathfrak{H}_2$. Then as we have seen above $E\pi(x)\phi = \{\pi(x)\phi(1)\}\psi = \phi(x)\psi$ $(x \varepsilon G)$. Since $\phi \neq 0$ it is clear that $\phi(x) \neq 0$ for some x and therefore $\psi \varepsilon \mathfrak{H}_2$. But this implies that $\mathfrak{H}_2 = \mathfrak{H}_1$ and therefore $\mathfrak{H}_2$ is irreducible.

Now let z be any element in $\mathfrak{B}$ of rank[3] zero. Then it is clear that if ϕ is a well-behaved element in $\mathfrak{H}$,

$$\pi(h)\pi(z)\phi = \pi(z)\pi(h)\phi \qquad (h \varepsilon A),$$

and therefore $E\pi(z)\phi = \pi(z)E\phi$. In particular

$$E\pi(z)\psi = \pi(z)E\psi = \pi(z)\psi.$$

But we know that $E\phi = \phi(1)\psi$ for every $\phi \varepsilon \mathfrak{H}$. Therefore

$$\pi(z)\psi = E\pi(z)\psi = \chi(z)\psi,$$

where $\chi(z)$ is the value of $\pi(z)\psi$ at 1. Now if z lies in the center of $\mathfrak{B}$, it follows that

$$\pi(z)\pi(x)\psi = \pi(x)\pi(z)\psi = \chi(z)\pi(x)\psi \qquad (x \varepsilon G).$$

Hence if V is the subspace of $\mathfrak{H}_1$ spanned by $\pi(x)\psi$ $(x \varepsilon G)$, $\pi(z)\phi = \chi(z)\phi$ for all $\phi \varepsilon V$. Since V consists of well-behaved elements and since V is

dense in $\mathfrak{H}_1$ the quasi-simplicity of the representation on $\mathfrak{H}_1$ now follows from[9] Lemma 32 of [5(b)].

LEMMA 12. *Suppose $\mathfrak{H} \neq \{0\}$ and π is a unitary representation. Then $\mathfrak{H}$ is irreducible under π.*

In view of Lemma 11 it is enough to prove that $\mathfrak{H}_1 = \mathfrak{H}$. Let $\mathfrak{H}_2$ be the orthogonal complement of $\mathfrak{H}_1$ in $\mathfrak{H}$. Since π is unitary, $\mathfrak{H}_2$ is invariant under π. Let ϕ be any element in $\mathfrak{H}_2$. Then we have seen that $E\pi(x)\phi = \phi(x)\psi$ $(x \varepsilon G)$. Therefore $\phi(x)\psi \varepsilon \mathfrak{H}_2 \cap \mathfrak{H}_1 = \{0\}$ and so $\phi(x) = 0$. This being true for every $x \varepsilon G$, it is clear that $\phi = 0$. This proves that $\mathfrak{H}_2 = \{0\}$ and therefore $\mathfrak{H}_1 = \mathfrak{H}$.

6. Computation of the function ψ. We shall now determine the function ψ explicitly. We know from Lemma 4 that $Q = P_c^- M_c P_c^+$ is an open submanifold of G_c and there exists a (unique) holomorphic mapping m of Q into M_c such that $q \varepsilon P_c^- m(q) P_c^+$ for every $q \varepsilon Q$. Let $\tilde{M}_c$ be the simply connected covering group of M_c and γ the natural homomorphism of $\tilde{M}_c$ onto M_c. Since $\bar{W} \subset Q$, $w \rightarrow m(\nu(w))$ is a holomorphic mapping of W into M_c. But W is simply connected and so there exists a (unique) holomorphic mapping $\tilde{m}$ of W into $\tilde{M}_c$ such that $\tilde{m}(1) = 1$ and $\gamma \circ \tilde{m} = m \circ \nu$. Now if we use the notation of [5(f), § 5] it is clear that $\mathfrak{m} = \mathfrak{g}' + \mathfrak{k}$. Hence if $\mathfrak{c}$ is the center of $\mathfrak{k}$, $\mathfrak{c}_+ = \mathfrak{c} \cap \mathfrak{g}_+$ is the center of $\mathfrak{m}$ and $\mathfrak{m}$ is the direct sum of $\mathfrak{c}_+$ and $\mathfrak{m}' = [\mathfrak{m}, \mathfrak{m}]$. Therefore $\tilde{M}_c$ being simply connected, there exists a (unique) holomorphic mapping Γ of W into $\mathfrak{c}_+$ such that $\tilde{m}(w) \varepsilon \tilde{M}_c' \exp \Gamma(w)$. Here $\tilde{M}_c'$ is the analytic subgroup of $\tilde{M}_c$ corresponding to $\mathfrak{m}'$.

LEMMA 13. *Let w be any element in W. Then*

$$\Gamma(l_{na}w) = \Gamma(a) + \Gamma(w), \quad \Gamma(r_u w) = \Gamma(w) + \Gamma(u), \quad \Gamma(ux) = \Gamma(u) + \Gamma(x)$$

[9] It was pointed out to me by Dixmier that the proof of Lemma 33 of [5(b)] is incorrect. Since $\tilde{M}$ is dense in $\tilde{\mathfrak{H}}$ only in the weak topology, one cannot conclude that $|(\tilde{\phi}, A\psi)| \leq |B|_N |\tilde{\phi}| |\psi|$ for all $\tilde{\phi} \varepsilon \tilde{\mathfrak{H}}$ and $\psi \varepsilon M$, knowing it to be true for $\tilde{\phi} \varepsilon \tilde{M}$ and $\psi \varepsilon M$. However suppose there exists a positive real number a such that every element ψ_0 in $\tilde{\mathfrak{H}}$ can be approximated arbitrarily well (in the weak topology) by elements $\psi \varepsilon M$ with $|\psi| \leq a |\psi_0|$. Then it follows immediately that $|(\tilde{\phi}, A\psi)| \leq a |B|_N |\tilde{\phi}| |\psi|$ for all $\tilde{\phi} \varepsilon \tilde{\mathfrak{H}}$ and $\psi \varepsilon M$ and therefore $|A|_M \leq a |B|_N$. Hence, in particular, A is bounded if B is bounded. On the other hand if we use the notation of [5(b), pp. 226-227], it is clear that if n is sufficiently large $|A_{f_n}\psi| \leq a |\psi|$ where $a = 2 \sup_{x \varepsilon \omega} |\pi(x)|$ and ω is a compact set outside which all f_n are zero. From this it follows that $|\tilde{A}_{f_n}| \leq a$ and therefore the subspace $\tilde{V}$ of $\tilde{\mathfrak{H}}$ does have the above additional property. The proof of Lemma 32 of [5(b)] now goes through without any further modification.

for $n \varepsilon N_c^-$, $a \varepsilon \tilde{A}_c$, $u \varepsilon K$ *and* $x \varepsilon G$. *Moreover if* $a = \exp H \varepsilon \tilde{A}_c$ $(H \varepsilon \mathfrak{H})$ *and* $H = H_+ + H'$ $(H_+ \varepsilon \mathfrak{c}_+, H' \varepsilon \mathfrak{h} \cap \mathfrak{m}')$ *then* $\Gamma(a) = H_+$.

Since $[\mathfrak{m}, \mathfrak{p}_-] \subset \mathfrak{p}_-$ and $[\mathfrak{m}, \mathfrak{p}_+] \subset \mathfrak{p}_+$, it follows that $M_cQ = QM_c = Q$ and $m(vq) = vm(q)$, $m(qv) = m(q)v$ $(v \varepsilon M_c, q \varepsilon Q)$. Moreover $\mathfrak{n}_- \subset \mathfrak{p}_- + \mathfrak{k}'$ $\subset \mathfrak{p}_- + \mathfrak{m}'$. Therefore $N_c^- \subset P_c^-\gamma(\tilde{M}_c')$. Similarly if $u \varepsilon K$, $m(q\bar{u}) = m(q)\bar{u}$ $(q \varepsilon Q)$ and $m(\bar{u}x) = \bar{u}m(x)$ $(x \varepsilon G)$. The first part of the lemma follows immediately from these facts. Now put $a_+ = \exp H_+$ and $a' = \exp H'$. Then $a = a_+a'$ and therefore $\Gamma(a) = \Gamma(a_+) + \Gamma(a')$. But since $a' \varepsilon \tilde{M}_c'$, $\Gamma(a') = 0$. On the other hand it is obvious that $\Gamma(a_+) = H_+$ and so the lemma is proved.

By a complex representation of G_c we mean a finite-dimensional representation such that the corresponding representation of $\mathfrak{g}$ is linear over C. Since G_c is simply connected we may identify the finite-dimensional representations of $\mathfrak{g}$ with the complex representations of G_c. If V is the representation space of such a representation π, it would be convenient to regard V as a Hilbert space in such a way that π becomes unitary on the subgroup U corresponding to $\mathfrak{u} = \mathfrak{k}_0 + (-1)^{\frac{1}{2}}\mathfrak{p}_0$. Since U is compact, this is always possible. Whenever we speak of a finite-dimensional representation of $\mathfrak{g}$ (or of a complex representation of G_c) we shall tacitly assume that such a Hilbert space structure has already been introduced in the representation space.

Now let Λ be the linear function of Section 5 and let $\alpha_1, \cdots, \alpha_l$ be a fundamental system of positive roots of $\mathfrak{g}$ (see Corollary 2 to Lemma 4 of [5(f)]). We assume that $\alpha_1, \cdots, \alpha_t$ are all the totally positive roots among these. Define a linear function Λ_0 on $\mathfrak{h}$ by the conditions $\Lambda_0(H_{\alpha_i}) = 0$ $1 \leqq i \leqq t$ and $\Lambda_0(H_{\alpha_i}) = \Lambda(H_{\alpha_i})$ $(t < i \leqq l)$. Then Λ_0 is a dominant integral function and therefore from Theorem 1 of [5(a)] there exists an irreducible representation σ of $\mathfrak{g}$ on the finite-dimensional space V with the highest weight Λ_0. Let ϕ_0 be a unit vector in V belonging to the weight Λ_0 and put $\lambda = \Lambda - \Lambda_0$.

LEMMA 14. *Let ξ be the holomorphic character of $\tilde{A}_c$ corresponding to Λ. Then the function ψ of Lemma 6 is given by the formula*

$$\psi(w) = (\phi_0, \sigma(\nu(w))\phi_0)e^{\lambda(\Gamma(w))} \qquad (w \varepsilon W).$$

First observe that since σ is unitary on U, the adjoint of the operator $\sigma(X)$ is $-\sigma(\tilde{\theta}(X))$ $(X \varepsilon \mathfrak{g})$. Hence $\sigma(\tilde{\theta}(z^{-1}))$ is the adjoint of $\sigma(z)$ $(z \varepsilon G_c)$. But $\tilde{\theta}(N_c^-) = N_c^+$ and since ϕ_0 belongs to the highest weight, $\sigma(n')\phi_0 = \phi_0$ if $n' \varepsilon N_c^+$. Moreover from Lemma 13, $\Gamma(l_nw) = \Gamma(w)$ $(n \varepsilon N_c^-)$. Therefore if ψ_0 denotes the expression on the right hand side of the above

equation, $\psi_0(l_n w)=\psi_0(w)$ $(n\,\varepsilon\, N_c^-)$. Now let $a=\exp H\,\varepsilon\,\tilde{A}_c$ $(H\,\varepsilon\,\mathfrak{h})$. Then if $\bar{w}=\nu(w)$ $(w\,\varepsilon\, W)$,

$$(\phi_0,\sigma(\nu(l_a w))\phi_0)=(\sigma(\bar{\theta}(\bar{a}^{-1}))\phi_0,\sigma(\bar{w})\phi_0)=e^{\Lambda_0(H)}(\phi_0,\sigma(\bar{w})\phi_0)$$

since Λ_0 is real. On the other hand it follows from Lemma 13 that $\lambda(\Gamma(l_a w))=\lambda(H_+)+\lambda(\Gamma(w))$ where $H=H_++H'$ $(H_+\,\varepsilon\,\mathfrak{c}_+, H'\,\varepsilon\,\mathfrak{h}\cap\mathfrak{m}')$. Therefore

$$\psi_0(l_a w)=\exp(\Lambda_0(H)+\lambda(H_+))\psi_0(w).$$

But $\Lambda(H)=\Lambda(H_+)+\Lambda(H')$ and it is obvious from Lemma 13 of [5(f)] that H' is a linear combination of H_{α_i} $(t<i\leqq l)$ so that $\lambda(H')=0$. Hence

$$\Lambda_0(H)+\lambda(H_+)=\Lambda_0(H)+\lambda(H)=\Lambda(H).$$

This proves that $\psi_0(l_a w)=\xi(a)\psi_0(w)$ and so $\psi_0\,\varepsilon\,\mathfrak{H}_\xi$. Moreover if $a\,\varepsilon\, A$,

$$\psi_0(r_a w)=(\phi_0,\sigma(\nu(w)\bar{a})\phi_0)\exp(\lambda(\Gamma(w))+\lambda(\Gamma(a)))=\psi_0(w)\xi(a)$$

since $\sigma(\bar{a})\phi_0 e^{\lambda(\Gamma(a))}=\xi(a)\phi_0$. Hence

$$\psi_0(w^{\bar{h}})=\psi_0(l_h r_{h^{-1}} w)=\xi(h)\xi(h^{-1})\psi_0(w)=\psi_0(w) \qquad (h\,\varepsilon\, A)$$

and therefore $\psi_0(w)=\int_{A_0}\psi_0(w^{\bar{h}})d\bar{h}=\psi_0(1)\psi(w)$ from Lemma 6. But $\psi_0(1)=|\phi_0|^2=1$ and so $\psi_0=\psi$.

We have still to show that the space $\mathfrak{H}$ $(\mu)\neq\{0\}$ for a suitable choice of μ. In order to do this we need some preliminary results on finite-dimensional representations of $\mathfrak{g}$ which we shall then apply to the above representation σ.

7. Some results on finite-dimensional representations. First we prove the following lemma.

LEMMA 15. *There exists an element $H\,\varepsilon\,\mathfrak{h}_0$ such that $\theta(X)=\exp(adH)X$ for all $X\,\varepsilon\,\mathfrak{g}$.*

Since $\theta(H)=H$ $(H\,\varepsilon\,\mathfrak{h})$ and $\theta^2(X)=X$ $(X\,\varepsilon\,\mathfrak{g})$, it is clear that $\theta(X_\alpha)=\pm X_\alpha$ for every root α. If $(\alpha_1,\cdots,\alpha_l)$ is a fundamental system of roots we can choose $H\,\varepsilon\,\mathfrak{h}$ such that $\theta(X_{\alpha_i})=e^{\alpha_i(H)}X_{\alpha_i}$ $1\leqq i\leqq l$. It is obvious that $\alpha_1(H),\cdots,\alpha_l(H)$ are all purely imaginary and therefore $H\,\varepsilon\,\mathfrak{h}_0$. Moreover since $[X_{\alpha_i},X_{-\alpha_i}]\,\varepsilon\,\mathfrak{h}$, it follows that $\theta(X_{-\alpha_i})=e^{-\alpha_i(H)}X_{\alpha_i}$. Therefore θ and $\exp(adH)$ are two automorphisms of $\mathfrak{g}$ which coincide on $\mathfrak{h}$ and also at X_{α_i}, $X_{-\alpha_i}$ $1\leqq i\leqq l$. But $\mathfrak{g}$ is the smallest subalgebra of itself containing $\mathfrak{h}$ and X_{α_i}, $X_{-\alpha_i}$ $1\leqq i\leqq l$ (see Lemmas 18 and 19 of [5(a)]) and so $\theta=\exp(adH)$.

LEMMA 16. *Let π be a representation of $\mathfrak{g}$ on a finite-dimensional space V. Then if ϕ is a vector belonging to some weight of π*

$$(\phi, \pi(z)\phi) = (\phi, \pi(\theta(z))\phi) \qquad (z \varepsilon G_c).$$

From Lemma 15 we can choose $h \varepsilon A_0$ such that $\theta(z) = hzh^{-1}$ for all $z \varepsilon G_c$. Then $(\phi, \pi(\theta(z))\phi) = (\pi(h^{-1})\phi, \pi(zh^{-1})\phi)$ since π is unitary on A_0. But since ϕ belongs to a weight of π, $\pi(h^{-1})\phi = c\phi$ where c is a unimodular complex number. Hence

$$(\phi, \pi(\theta(z))\phi) = |c|^2(\phi, \pi(z)\phi) = (\phi, \pi(z)\phi).$$

Now ϕ being as above, we propose to study the growth of the function $F(X) = (\phi, \pi(\exp X)\phi)$ $(X \varepsilon \mathfrak{p}_0)$ at infinity. Let X be a fixed element in $\mathfrak{p}_0$. Since $\theta(X) = -X$, $\pi(X)$ is a self-adjoint operator on V and therefore we can choose an orthonormal base $(\phi_1, \cdots, \phi_d)$ for V such that $\pi(X)\phi_i = \lambda_i\phi_i$ $(\lambda_i \varepsilon R,\ i = 1, \cdots, d)$. Then if $\phi = \sum_i a_i\phi_i$ $(a_i \varepsilon C)$,

$$F(X) = \sum_i |a_i|^2 e^{\lambda_i}.$$

Similarly $F(-X) = \sum_i |a_i|^2 e^{-\lambda_i}$. But since $\theta(X) = -X$, it follows from Lemma 16 that $F(X) = F(-X)$ and therefore $F(X) = \sum_i |a_i|^2 \cosh \lambda_i$. Now consider the function

$$c(t) = (\cosh t - 1)/t^2 = \sum_{n=1}^{\infty} t^{2n-2}/(2n)! \qquad (t \varepsilon R).$$

Then $c(t) = c(-t) \geqq 0$ and $c(t)$ increases with t for $t \geqq 0$. Also

$$F(X) = \sum_i |a_i|^2 + \sum_i |a_i\lambda_i|^2 c(\lambda_i) = 1 + \sum_i |a_i\lambda_i|^2 c(|\lambda_i|)$$

if we assume that $|\phi| = 1$. On the other hand $|\pi(X)\phi|^2 = \sum_i |a_i\lambda_i|^2$ and therefore $d^{-\frac{1}{2}}|\pi(X)\phi| \leqq \max_i |a_i\lambda_i|$. Let j be an index such that $|a_j\lambda_j| = \max_i |a_i\lambda_i|$. Then

$$\begin{aligned} F(X) &\geqq 1 + |a_j\lambda_j|^2 c(|\lambda_j|) \geqq 1 + |a_j\lambda_j|^2 c(|a_j\lambda_j|) \\ &\geqq 1 + d^{-1}|\pi(X)\phi|^2 c(d^{-\frac{1}{2}}|\pi(X)\phi|) = \cosh(d^{-\frac{1}{2}}|\pi(X)\phi|). \end{aligned}$$

Therefore we have the following result.

LEMMA 17. *Let π and ϕ be as in Lemma 16. Then if $|\phi| = 1$ and $X \varepsilon \mathfrak{p}_0$,*

$$(\phi, \pi(\exp X)\phi) \geqq \cosh(d^{-\frac{1}{2}}|\pi(X)\phi|)$$

where $d = \dim V$.

On the other hand we have the following result concerning $|\pi(X)\phi|$.

LEMMA 18. *Suppose ϕ belongs to the highest weight λ of π. Then if $\lambda(H_\beta)\neq 0$ for every noncompact root β, $|\pi(X)\phi|^2$ $(X\,\varepsilon\,\mathfrak{p}_0)$ is a positive definite quadratic form on $\mathfrak{p}_0$.*

Let X be an element in $\mathfrak{p}_0$. Then if we choose $X_\beta, X_{-\beta}$ as in [5(f), § 4], we can write $X=\sum_\beta (c_\beta X_\beta+\bar{c}_\beta X_{-\beta})$ where β runs over all noncompact positive roots and the bar denotes complex conjugate. Since ϕ belongs to the highest weight $\pi(X_\beta)\phi=0$. Hence $\pi(X)\phi=\sum_\beta \bar{c}_\beta\pi(X_{-\beta})\phi$. On the other hand if $\lambda(H_\beta)\neq 0$ we know (Lemma 1 of [5(f)]) that $\pi(X_{-\beta})\phi\neq 0$ and it belongs to the weight $\lambda-\beta$. Since nonzero vectors belonging to distinct weights are linearly independent, we conclude that $\pi(X)\phi\neq 0$ unless $\bar{c}_\beta=0$ for all β. This proves our assertion.

In Section 2 we have defined a (real) analytic mapping $z\rightarrow H(z)$ of G_c into $(-1)^{\frac{1}{2}}\mathfrak{h}_0$ such that $z\,\varepsilon\,U(\exp H(z))N_c^+$ $(z\,\varepsilon\,G_c)$. We shall say that a linear function λ on $\mathfrak{h}$ is *completely positive* if $\lambda(H_\alpha)$ is real and nonnegative for every positive root α.

LEMMA 19. *If λ is a completely positive linear function on $\mathfrak{h}$ then $\lambda(H(x))\geqq 0$ for all $x\,\varepsilon\,G$. Moreover for any $X\,\varepsilon\,\mathfrak{p}_0$ the function $\lambda(H(\exp tX))$ $(t\,\varepsilon\,R)$ is non-decreasing for $t\geqq 0$. Finally $H(p)=H(p^{-1})$ if $p\,\varepsilon\exp\mathfrak{p}_0$.*

Let $\alpha_1,\cdots,\alpha_l$ be a fundamental system of positive roots. Define real linear functions $\Lambda_1,\cdots,\Lambda_l$ as follows: $\Lambda_i(H_{\alpha_j})=\delta_{ij}$ $1\leqq i,j\leqq l$ where $\delta_{ij}=1$ or 0 according as $i=j$ or not. Then $\lambda=\lambda_1\Lambda_1+\cdots+\lambda_l\Lambda_l$ where $\lambda_i=\Lambda(H_{\alpha_i})$ are nonnegative real numbers. Since Λ_i are dominant integral functions (see [5(a), p. 30]), it is clearly sufficient to prove the lemma under the assumption that λ is such a function. Let π be a finite-dimensional irreducible representation of $\mathfrak{g}$ on a space V with the highest weight λ [5(a), Theorem 1] and let ϕ be a unit vector in V belonging to the weight λ. Then if $x=uhn$ $(x\,\varepsilon\,G_0, u\,\varepsilon\,U, h\,\varepsilon\,A_+, n\,\varepsilon\,N_c^+)$ it is clear that

$$|\pi(x)\phi|=|\pi(uh)\phi|=|\pi(h)\phi|=e^{\lambda(H(x))}.$$

On the other hand $x=vp$ $(v\,\varepsilon\,K_0, p\,\varepsilon\exp\mathfrak{p}_0)$ and therefore

$$|\pi(x)\phi|^2=|\pi(p)\phi|^2=(\phi,\pi(p^2)\phi),$$

since $\pi(p)$ is self-adjoint. But it follows from Lemma 17 that $(\phi,\pi(p^2)\phi)\geqq 1$ and therefore $e^{\lambda(H(x))}=|\pi(x)\phi|\geqq 1$. This proves that $\lambda(H(x))\geqq 0$. Moreover if $X\,\varepsilon\,\mathfrak{p}$, $\pi(X)$ is self-adjoint and therefore

$$\exp 2\lambda(H(\exp tX)) = |\pi(\exp tX)\phi|^2$$
$$= (\phi, \pi(\exp 2tX)\phi) = (\phi, \pi(\exp(-2tX)\phi) \qquad (t \,\varepsilon\, R)$$

from Lemma 16. Hence

$$\exp 2\lambda(H(\exp tX)) = \tfrac{1}{2}(\phi, \pi(\exp 2tX)\phi) + \tfrac{1}{2}(\phi, \pi(\exp(-2tX)\phi)$$
$$= \sum_{n \geqq 0} (2t)^{2n} |(\pi(X))^n \phi|^2 / 2n!.$$

Since $\lambda(H(\exp tX))$ is real and since only even powers of t occur and all the coefficients of the series are nonnegative, the second assertion of the lemma is now obvious. Also it is clear that $\lambda(H(p)) = \lambda(H(p^{-1}))$ for $p \,\varepsilon \exp \mathfrak{p}_0$. This is true in particular for $\lambda = \Lambda_i$ $i = 1, \cdots, l$. Since $\Lambda_1, \cdots, \Lambda_l$ is a base for the space of all linear functions on $\mathfrak{h}$, we conclude that $H(p) = H(p^{-1})$.

Let $B(X, Y) = sp(adXadY)$ $(X, Y \,\varepsilon\, \mathfrak{g})$. Then $-B(\tilde{\theta}(X), X)$ is a positive definite Hermitian form on $\mathfrak{g}$ which we denote by $\|X\|^2$. We now regard $\mathfrak{g}$ as a Helibert space under the corresponding norm $\|\cdot\|$. By combining Lemmas 17, 18 and 19 we can get the following stronger result.

LEMMA 20. *Suppose λ is completely positive and $\lambda(H_\beta) \neq 0$ for every noncompact root β. Then there exists a positive real number c such that $\lambda(H(\exp X)) \geqq c\|X\|$ for all $X \,\varepsilon\, \mathfrak{p}_0$ lying outside some bounded set.*

First suppose $\lambda(H_{\alpha_i})$ $1 \leqq i \leqq l$ are all integers so that λ is a dominant integral function. We define π, V and ϕ as in the proof of Lemma 19. Then from Lemma 17,

$$(\phi, \pi(\exp 2X)\phi) \geqq \cosh(2d^{-\frac{1}{2}}|\pi(X)\phi|)$$

where $d = \dim V$. But we have seen above that if $p = \exp X$,

$$(\phi, \pi(\exp 2X)\phi) = |\pi(p)\phi|^2 = e^{2\lambda(H(p))}.$$

Hence

$$\lambda(H(p)) \geqq \tfrac{1}{2}\log(\cosh(2d^{-\frac{1}{2}}|\pi(X)\phi|)) \geqq d^{-\frac{1}{2}}|\pi(X)\phi| - \tfrac{1}{2}\log 2.$$

On the other hand we know from Lemma 18 that $|\pi(X)\phi|^2$ $(X \,\varepsilon\, \mathfrak{p}_0)$ is a positive definite quadratic form on $\mathfrak{p}_0$. Hence if c_0 is the least possible value of $|\pi(X)\phi|$ for all $X \,\varepsilon\, \mathfrak{p}_0$ with $\|X\| = 1$, $c_0 > 0$ and $|\pi(X)\phi| \geqq c_0\|X\|$ $(X \,\varepsilon\, \mathfrak{p}_0)$. Therefore

$$\lambda(H(\exp X)) \geqq c_0 d^{-\frac{1}{2}}\|X\| - \tfrac{1}{2}\log 2 \geqq \tfrac{1}{2}c_0 d^{-\frac{1}{2}}\|X\|$$

provided $\|X\| \geqq c_0^{-1} d^{\frac{1}{2}} \log 2$ $(X \,\varepsilon\, \mathfrak{p}_0)$. Hence we may take $c = \tfrac{1}{2}c_0 d^{-\frac{1}{2}}$.

Now we come to the general case. Define $\Lambda_1, \cdots, \Lambda_l$ as in the proof of Lemma 19. Then $\lambda = \lambda_1\Lambda_1 + \cdots + \lambda_l\Lambda_l$ where $\lambda_i = \lambda(H_{\alpha_i})$ are nonnegative real numbers. Choose a positive number r so large that if $\lambda_i \neq 0$, $r\lambda_i \geqq 1$ $(1 \leqq i \leqq l)$ and put $\lambda_0 = a_1\Lambda_1 + \cdots + a_l\Lambda_l$ where $a_i = 0$ or 1 according as $\lambda_i = 0$ or not. Then $r\lambda - \lambda_0$ is completely positive and λ_0 is a dominant integral function. Let α be a positive root. Then $H_\alpha = b_1 H_{\alpha_1} + \cdots + b_l H_{\alpha_l}$ $(b_i \varepsilon R, b_i \geqq 0)$. Suppose $b_i > 0$ $(1 \leqq i \leqq m)$ and $b_i = 0$ $(m < i \leqq l)$. Then $\lambda(H_\alpha) = b_1\lambda_1 + \cdots + b_m\lambda_m$ and $\lambda_0(H_\alpha) = b_1 a_1 + \cdots + b_m a_m$. Since λ_i, a_i are all nonnegative and since $\lambda_i = 0$ if and only if $a_i = 0$, it is clear that $\lambda(H_\alpha) = 0$ if and only if $\lambda_0(H_\alpha) = 0$. Therefore $\lambda_0(H_\beta) \neq 0$ for any noncompact positive root β and so, by the above proof, there exists a positive number c' such that $\lambda_0(H(\exp X)) \geqq c' \| X \|$ $(X \varepsilon \mathfrak{p}_0)$ provided that $\| X \|$ is sufficiently large. However since $r\lambda - \lambda_0$ is completely positive

$$r\lambda(H(\exp X)) \geqq \lambda_0(H(\exp X))$$

(Lemma 19) and therefore

$$\lambda(H(\exp X)) \geqq c' \| X \| / r \qquad (X \varepsilon \mathfrak{p}_0)$$

if $\| X \|$ is sufficiently large.

On the other hand it is quite easy to obtain a result in the opposite direction.

LEMMA 21. *Let λ be a linear function on $\mathfrak{h}$. Then there exists a real number c' such that $| \lambda(H(\exp X)) | \leqq c' \| X \|$ for all $X \varepsilon \mathfrak{p}_0$.*

Put $\lambda_i = \Lambda(H_{\alpha_i})$ $i = 1, 2, \cdots, l$. Then $\lambda = \lambda_1\Lambda_1 + \cdots + \lambda_l\Lambda_l$ and $| \lambda(H) | \leqq \sum_i | \lambda_i | \, | \Lambda_i(H) |$ $(H \varepsilon \mathfrak{h})$. Therefore it is obviously sufficient to prove the lemma under the assumption that λ is a dominant integral function. Define π, V and ϕ as above. Then if $p = \exp X$ $(X \varepsilon \mathfrak{p}_0)$,

$$e^{\lambda(H(p))} = | \pi(p)\phi | \leqq | \pi(p) | \leqq e^{|\pi(X)|}$$

where $| T |$ denotes, as usual, the bound of an operator T. Therefore from Lemma 19,

$$| \lambda(H(p)) | = \lambda(H(p)) \leqq | \pi(X) | \leqq \{ sp(\pi(X))^2 \}^{\frac{1}{2}}$$

since $\pi(X)$ is self-adjoint. If a' is the maximum value of $sp(\pi(X))^2$ for all $X \varepsilon \mathfrak{p}_0$ with $\| X \| = 1$, it is clear that $sp(\pi(X))^2 \leqq a' \| X \|^2$ for every X in $\mathfrak{p}_0$ and therefore $| \lambda(H(\exp X)) | \leqq c' \| X \|$ where $c' = (a')^{\frac{1}{2}}$.

We shall also need another similar result. Let ρ be defined as in Lemma 3.

LEMMA 22. *Let π be a representation of $\mathfrak{g}$ on a finite-dimensional space V. Then there exist positive real numbers a, b such that*

$$|\pi(x^{-1})\phi| \leqq be^{a\rho(H(x))}|\phi|$$

for every $\phi \varepsilon V$ and $x \varepsilon G_0$.

If $x = up$ $(u \varepsilon K_0, p \varepsilon \exp \mathfrak{p}_0)$, $\pi(x^{-1})\phi = \pi(p^{-1})\phi'$ where $\phi' = \pi(u)\phi$. Also $H(x) = H(p)$ and $|\phi'| = |\phi|$. So it is enough to consider the case when $u = 1$. Let $p = \exp X$ $(X \varepsilon \mathfrak{p}_0)$. Then

$$|\pi(p^{-1})\phi| \leqq |\pi(p^{-1})|\,|\phi| \leqq e^{|\pi(X)|}|\phi|.$$

But as we have just seen above, there exists a real number a_1 such that $|\pi(X)| \leqq a_1 \|X\|$ for all $X \varepsilon \mathfrak{p}_0$. On the other hand from Lemma 20 we can find positive real numbers a_2 and M such that [10]

$$\rho(H(\exp X)) \geqq a_2 \|X\|$$

for all $X \varepsilon \mathfrak{p}_0$ for which $\|X\| \geqq M$. Let b be an upper bound for $|\pi(\exp(-X))|$ when $\|X\| \leqq M$ $(X \varepsilon \mathfrak{p}_0)$. Then if $a = a_1/a_2$ it is obvious that $|\pi(x^{-1})\phi| \leqq be^{a\rho(H(x))}$ for all $\phi \varepsilon V$ and $x \varepsilon G$.

The following lemma describes some properties of the mapping $x \rightarrow \Gamma(x)$ $(x \varepsilon G)$.

LEMMA 23. *Let λ be a real linear function on $\mathfrak{h}$. Then if $u \varepsilon K$, the real part of $\lambda(\Gamma(u))$ is zero. Moreover if $\lambda(H_\alpha) = 0$ for every positive root α which is not totally positive,*

$$\lambda(\Gamma(\exp X)) = 2\lambda(H(\exp \tfrac{1}{2}X)) \qquad (X \varepsilon \mathfrak{p}_0).$$

We know that $\mathfrak{h}$ is the direct sum of $\mathfrak{c}_+$ and $\mathfrak{h} \cap \mathfrak{m}'$ (in the notation of Section 6) and if α is a positive root which is not totally positive, $H_\alpha \varepsilon \mathfrak{h} \cap \mathfrak{m}'$. Without affecting its values on $\mathfrak{c}_+$ we can replace λ by a linear function which vanishes identically on $\mathfrak{h} \cap \mathfrak{m}'$ and therefore assume that $\lambda(H_\alpha) = 0$ for all positive roots α which are not totally positive. Then $\lambda = \lambda_1\Lambda_1 + \cdots + \lambda_t\Lambda_t$ (in the notation of the proof of Lemma 14) where $\lambda_1, \cdots, \lambda_t$ are real numbers. Obviously it would be enough to prove the lemma for $\lambda = \Lambda_i$ $i = 1, \cdots, t$. Hence we may assume that λ is a dominant integral function. Let π be an irreducible representation of $\mathfrak{g}$ on a finite-dimensional space V with the highest weight λ and let ϕ be a unit vector in V belonging to the weight λ. Then if β is a positive root which is not totally positive, $\lambda(H_\beta) = 0$ and therefore from Lemma 1 of [5(f)] $\pi(X_{-\beta})\phi = 0$. Hence it is clear that

[10] Here we have to make use of the fact (see Weyl [9]) that $\rho(H_\alpha) > 0$ for every positive root α.

$\pi(\mathfrak{m}'+\mathfrak{p}_+)\phi=\{0\}$. On the other hand we have seen in Section 6 that $\bar{x}=qm(\bar{x})p$ $(x \varepsilon G)$ where $q \varepsilon P_c^-$, $p \varepsilon P_c^+$. Therefore

$$(\phi,\pi(\bar{x})\phi)=(\pi(\theta(q^{-1}))\phi,\pi(m(\bar{x}))\phi)=(\phi,\pi(m(\bar{x})\phi)$$

since $\bar{\theta}(q^{-1}) \varepsilon P_c^+$. Moreover since $\pi(\mathfrak{m}')\phi=\{0\}$, it is obvious that

$$\pi(m(\bar{x}))\phi=e^{\lambda(\Gamma(x))}\phi$$

and therefore $(\phi,\pi(\bar{x})\phi)=e^{\lambda(\Gamma(x))}$ $(x \varepsilon G)$. Now if $u \varepsilon K$, $\bar{u} \varepsilon M_c$ and therefore $\pi(\bar{u})\phi=e^{\lambda(\Gamma(u))}\phi$. But since $\pi(\bar{u})$ is unitary, $\lambda(\Gamma(u))$ must be purely imaginary. On the other hand if $x=p^2$ where $p=\exp\frac{1}{2}X$ $(X \varepsilon \mathfrak{p}_0)$, the above equation gives $|\pi(\bar{p})\phi|^2=e^{\lambda(\Gamma(p^2))}$. But we have seen that $|\pi(\bar{p})\phi| = e^{\lambda(H(\bar{p}))}$. Hence

$$\lambda(\Gamma(\exp X))-2\lambda(H(\exp\tfrac{1}{2}X))$$

must be an integral multiple of $2\pi(-1)^{\frac{1}{2}}$. However it is a continuous function of X and it is zero at $X=0$. Therefore since $\mathfrak{p}_0$ is connected it must be everywhere zero on $\mathfrak{p}_0$. This proves the lemma.

COROLLARY. *If $u \varepsilon K$, $\Gamma(u)$ lies in $\mathfrak{h}_0$. Moreover*

$$\Gamma(\exp X)-\tfrac{1}{2}H(\exp\tfrac{1}{2}X)\ \varepsilon\ \mathfrak{h}\cap\mathfrak{m}'$$

for all $X \varepsilon \mathfrak{p}_0$.

This is an immediate consequence of the above lemma.

8. Proof of the existence of representations. In order to show that the space $\mathfrak{H}_\Lambda(\mu)$ of Lemma 9 is not zero for some μ, it is enough to find a function μ on G_0, satisfying the conditions of Section 6 and such that

$$\int_{G_0}|\psi(x)|^2\mu(\bar{x})d\bar{x}<\infty$$

where ψ is the function of Lemma 14. But $|\psi(x^{-1})| \leqq |\sigma(\bar{x}^{-1})\phi_0| e^{\Re(\lambda(\Gamma(x^{-1})))}$, (in the notation of Lemma 14) where $\Re c$ denotes the real part of a complex number c. On the other hand we know from Lemma 29 that

$$|\sigma(\bar{x}^{-1})\phi_0| \leqq be^{a\rho(H(\bar{x}))} \qquad (x \varepsilon G)$$

for suitable real numbers a and b. If $x=up$ $(u \varepsilon K,\ p=\exp X,\ X \varepsilon \mathfrak{p}_0)$ $\Gamma(x^{-1})=\Gamma(p^{-1})+\Gamma(u^{-1})$ (Lemma 13) and therefore

$$\Re(\lambda(\Gamma(x^{-1})))=\lambda(\Gamma(p^{-1}))=2\lambda(H(\exp(-\tfrac{1}{2}X)))=2\lambda(H(\exp\tfrac{1}{2}X))$$

from Lemmas 23 and 19. It is obvious[10] that $r'\rho-2\lambda$ is completely positive for a suitable real $r' \geqq 0$. Then

$$2\lambda(H(\exp\tfrac{1}{2}X)) \leqq r'\rho(H(\exp\tfrac{1}{2}X)) \leqq r'\rho(H(\exp X))=r'\rho(H(\bar{x}))$$

from Lemma 19. Hence $|\psi(x^{-1})| \leqq be^{(a+r')\rho(H(\bar{x}))}$ $(x \in G)$. Now put $\mu(\bar{x}) = e^{-r\rho(H(\bar{x}^{-1}))}$ $(\bar{x} \in G_0)$ where $r \geqq 2a + 2r' + 4$. Then μ is a continuous function on G_0 which is everywhere positive and

$$\int_{G_0} |\psi(x)|^2 \mu(\bar{x}) d\bar{x} = \int_{G_0} |\psi(x^{-1})|^2 \mu(\bar{x}^{-1}) d\bar{x} \leqq b^2 \int_{G_0} e^{-4\rho(H(\bar{x}))} d\bar{x} < \infty$$

from Lemma 3. Moreover since ρ is a dominant integral function (see Weyl [9]), there exists a finite-dimensional irreducible representation τ of $\mathfrak{g}$ with the highest weight ρ. Let ζ be a unit vector in the representation space belonging to the weight ρ. Then $|\tau(\bar{x}^{-1})\zeta| = e^{\rho(H(\bar{x}^{-1}))}$ $(\bar{x} \in G_0)$ and therefore, if $\bar{x}, \bar{y} \in G_0$,

$$e^{\rho(H(\bar{y}^{-1}\bar{x}^{-1}))} = |\tau(\bar{y}^{-1}\bar{x}^{-1})\zeta| \leqq |\tau(\bar{y}^{-1})| \, e^{\rho(H(\bar{x}^{-1}))}.$$

This proves that $\mu(\bar{x}\bar{y}) \leqq |\tau(\bar{y})|^r \mu(\bar{x})$ and since $|\tau(\bar{y})|$ is bounded on every compact set in G_0, μ fulfills all the conditions of Section 6. Hence $\psi \in \mathfrak{H}_\Lambda(\mu)$ and so $\mathfrak{H}_\Lambda(\mu) \neq \{0\}$.

We can now summarise our results in the following theorem.

THEOREM 2. *Let Λ be a real linear function on $\mathfrak{h}$ such that $\Lambda(H_\alpha)$ is a nonnegative integer for every positive root α which is not totally positive. Let ξ denote the character of $\tilde{A}_c$ defined by $\xi(\exp H) = e^{\Lambda(H)}$ $(H \in \mathfrak{h})$ and let μ be a measurable function on G_0 which is everywhere positive and such that $\sup_{\bar{x} \in G_0} \mu(\bar{x}\bar{y})/\mu(\bar{x})$ $(\bar{y} \in G_0)$ remains bounded on every compact subset of G_0. Let $\mathfrak{H}_\Lambda(\mu)$ denote the Hilbert space of all holomorphic functions f on W which fulfill the following two conditions:*

(1) $f(l_n a w) = \xi(a) f(w)$ $(a \in \tilde{A}_c, n \in N_c^-, w \in W)$

(2) $\|f\|^2 = \int_{G_0} |f(x)|^2 \mu(\bar{x}) d\bar{x} < \infty$.

Then if $\mathfrak{H}_\Lambda(\mu) \neq \{0\}$ there exists a unique function ψ in $\mathfrak{H}_\Lambda(\mu)$ such that $\psi(1) = 1$ and $\psi(w^{\bar{h}}) = \psi(w)$ for all $\bar{h} \in A_0$ and $w \in W$. Put $\psi_x(w) = \psi(r_x w)$, $(x \in G, w \in W)$ and let $\mathfrak{H}$ denote the smallest closed subspace of $\mathfrak{H}_\Lambda(\mu)$ containing ψ_x for all $x \in G$. Then we can define a quasi-simple irreducible representation π of G on $\mathfrak{H}$ by the rule

$$\pi(x)\phi(w) = \phi(r_x w) \qquad (x \in G, \phi \in \mathfrak{H}, w \in W).$$

This representation is infinitesimally equivalent to the representation π_Λ of [5(f), § 6]. *Finally, it is always possible to choose μ in such a way that $\mathfrak{H}_\Lambda(\mu) \neq \{0\}$.*

The infinitesimal equivalence follows immediately from Lemmas 8 and 10, Theorem 2 of [5(f)] and Theorem 5 of [5(b)].

COROLLARY. *Suppose* $\mu=1$ *and* $\mathfrak{H}_\Lambda(\mu)\neq\{0\}$. *Then* $\mathfrak{H}=\mathfrak{H}_\Lambda(\mu)$, π *is unitary and*

$$(\psi,\pi(x)\psi)=\psi(x)\|\psi\|^2 \qquad (x\,\varepsilon\, G).$$

It is obvious that if $f\,\varepsilon\,\mathfrak{H}_\Lambda(\mu)$,

$$\int_{G_0}|f(xy)|^2 d\bar{x}=\|f\|^2 \qquad (y\,\varepsilon\, G),$$

and therefore from Lemma 12, $\mathfrak{H}=\mathfrak{H}_\Lambda(\mu)$ and π is unitary. Moreover

$$\int_{A_0}\psi(y^{\bar{h}}x)\,d\bar{h}=\psi(x)\psi(y) \qquad (x,y\,\varepsilon\, G)$$

from Lemma 6. Hence [11]

$$\begin{aligned}\psi(x)\|\psi\|^2&=\int_{G_0}\{(\operatorname{conj}\psi(y))\int_{A_0}\psi(y^{\bar{h}}x)\,d\bar{h}\}\,d\bar{y}\\ &=\int_{A_0}d\bar{h}\int_{G_0}(\operatorname{conj}\psi(y))\psi(yx)\,d\bar{y}=(\psi,\pi(x)\psi)\end{aligned}$$

by Fubini's Theorem.

9. Unitary representations. Our next object is to look for unitary representations among those constructed above. However the result which we give below is not quite as strong as Theorem 3 of [5(f)] which was stated there without proof. We have to postpone its proof to another paper since it requires a deeper study of the representations.

THEOREM 3. *Let* $(\alpha_1,\cdots,\alpha_l)$ *be a fundamental system of positive roots and suppose* $(\alpha_1,\cdots,\alpha_t)$ *are all the totally positive roots in this system. Let* λ_i $(t<i\leqq l)$ *be given nonnegative integers such that* $\lambda_i=0$ *if* α_i *is not a root of*[3] $\mathfrak{g}_+$. *Then we can find a real number* c *with the following property. If* Λ *is a real linear function on* $\mathfrak{h}$ *and* $\Lambda(H_{\alpha_i})\leqq c$ $(1\leqq i\leqq t)$, $\Lambda(H_{\alpha_i})=\lambda_i$ $(t<i\leqq l)$, *then*[3] π_Λ *is infinitesimally unitary.*

Let Λ be a real linear function such that $\Lambda(H_{\alpha_i})=\lambda_i$ $(t<i\leqq l)$. Then $\Lambda(H_\alpha)=0$ for every root α of $\mathfrak{g}'$ (see [5(f), § 5]) and therefore $\pi_\Lambda(X)=0$ if $X\,\varepsilon\,\mathfrak{g}'$ (Lemma 19 of [5(f)]). Since $\mathfrak{g}_0=\mathfrak{g}_+\cap\mathfrak{g}_0+\mathfrak{g}'\cap\mathfrak{g}_0$ we can now restrict our attention to $\mathfrak{g}_+\cap\mathfrak{g}_0$. Hence without any essential loss of generality, we may asume that $\mathfrak{g}'=\{0\}$ and so $\mathfrak{g}=\mathfrak{g}_+$. Then it follows

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[11] conj c denotes the complex conjugate of a number $c\,\varepsilon\, C$.

from Lemma 13 of [5(f)] that α_i $(t < i \leqq l)$ are all compact. Define a linear function λ on $\mathfrak{h}$ by the conditions $\lambda(H_{\alpha_i}) = \Lambda(H_{\alpha_i})$ $1 \leqq i \leqq t$ and $\lambda(H_{\alpha_i}) = 0$ $t < i \leqq l$. Then λ is real and $\Lambda_0 = \Lambda - \lambda$ is a dominant integral function on $\mathfrak{h}$ and

$$\psi(x) = (\phi_0, \sigma(\bar{x})\phi_0)e^{\lambda(\Gamma(x))} \qquad (x \in G)$$

in the notation of Lemma 14. Now consider the integral

$$\int_{G_0} |\psi(x)|^2 \, d\bar{x}.$$

We know that

$$|\psi(x)| \leqq |\sigma(\bar{x})\phi_0| \, e^{\Re(\lambda(\Gamma(x)))} = \exp(\Lambda_0(H(\bar{x})) + \Re(\lambda(\Gamma(x))).$$

But if $x = u \exp X$ $(u \in K, X \in \mathfrak{p}_0)$

$$\Re(\lambda(\Gamma(x))) = \lambda(\Gamma(\exp X)) = 2\lambda(H(\exp \tfrac{1}{2}X))$$

from Lemmas 13 and 23. Now define Λ_i $1 \leqq i \leqq l$ as in the proof of Lemma 19. Then $\lambda_0 = \Lambda_1 + \cdots + \Lambda_t$ is a completely positive linear function and it follows from Lemma 13 of [5(f)] that $\lambda_0(H_\beta) = 1$ for every totally positive root. Since $\mathfrak{g} = \mathfrak{g}_+$, every noncompact positive root is totally positive and therefore Lemma 20 is applicable to λ_0. Hence there exists a positive number a such that

$$\lambda_0(H(\exp X)) \geqq a \, \| X \|,$$

for all $X \in \mathfrak{p}_0$ lying outside some bounded set. On the other hand from Lemma 21 we can find a real constant $b \geqq 0$ such that

$$\Lambda_0(H(\exp X)) + 2\rho(H(\exp X)) \leqq b \, \| X \| \qquad (X \in \mathfrak{p}_0).$$

Hence $\Lambda_0(H(\exp X)) + 2\rho(H(\exp X)) \leqq (2b/a)\lambda_0(H(\exp \tfrac{1}{2}X))$, for all $X \in \mathfrak{p}_0$ with a sufficiently large value of $\| X \|$. We note that b depends only on Λ_0 and therefore only on the integers λ_i $(t < i \leqq l)$. Now put $c = -b/a$. Then if $\Lambda(H_{\alpha_i}) \leqq c$ $1 \leqq i \leqq t$, $-\lambda + c\lambda_0$ is completely positive and therefore from Lemma 19

$$-2\lambda(H(\exp \tfrac{1}{2}X)) \geqq -2c\lambda_0(H(\exp \tfrac{1}{2}X)) \geqq \Lambda_0(H(\exp X)) + 2\rho(H(\exp X)),$$

if $\| X \|$ is sufficiently large $(X \in \mathfrak{p}_0)$. Hence

$$\exp\{\Lambda_0(H(\exp X)) + 2\rho(H(\exp X)) + \lambda(\Gamma(\exp X))\}$$

is bounded on $\mathfrak{p}_0$ and if M is an upper bound for it, it is obvious that

$$|\psi(x)| \leqq M e^{-2\rho(H(\bar{x}))} \qquad (x \in G).$$

Therefore $\int_{G_0} |\psi(x)|^2 d\bar{x} \leqq M^2 \int_{G_0} e^{-4\rho(H(\bar{x}))} d\bar{x} < \infty$ from Lemma 3. This proves that the space $\mathfrak{H}_\Lambda = \mathfrak{H}_\Lambda(1)$ (corresponding to the function $\mu = 1$) is not zero if $\Lambda(H_{\alpha_i}) \leqq c$ $(1 \leqq i \leqq t)$ and our assertion now follows immediately from Theorem 2 and its corollary.

10. A result on characters. Let Λ_0 be a linear function on $\mathfrak{h}$ satisfying the conditions of Theorem 2. We denote by T_{Λ_0} the character [5(c)] of the quasi-simple irreducible representation of G defined in Theorem 2 corresponding to Λ_0. Since two infinitesimally equivalent quasi-simple irreducible representations have the same character (see [5(c), § 7]), T_{Λ_0} is independent of the choice of μ in Theorem 2 and so it is completely determined by Λ_0. We shall now try to obtain some information about this character under suitable assumptions on Λ_0.

Let $\mathfrak{H}$ be a Hilbert space and Q a bounded operator on $\mathfrak{H}$. We say that Q is *summable* if there exists a complete orthonormal set $(\psi_j)_{j \varepsilon J}$ in $\mathfrak{H}$ and a regular operator B such that $\sum_{i,j \varepsilon J} | q_{ij} | < \infty$ where $q_{ij} = (\psi_i, BQB^{-1}\psi_j)$. Let $C_c^\infty(A)$ denote the set of all complex-valued functions on A which are everywhere indefinitely differentiable and which vanish outside a compact set. Since K is simply connected,[4] there exists (see Weyl [9]) an analytic function Δ_k on A such that

$$\Delta_k(\exp H) = \prod_{\alpha \varepsilon P_k} (e^{\alpha(H)/2} - e^{-\alpha(H)/2}) \qquad (H \varepsilon \mathfrak{h}_0)$$

where P_k is the set of all positive compact roots of $\mathfrak{g}$. As before we write $y^{\bar{x}} = xyx^{-1}$ $(x, y \varepsilon G)$. Let dh and $d\bar{u}$ denote the Haar measures on A and K_0 respectively. (We assume that $\int_{K_0} d\bar{u} = 1$). Then we have the following lemma.

LEMMA 24. *Let π be a quasi-simple irreducible representation of G on $\mathfrak{H}$. Then if $f \varepsilon C_c^\infty(A)$, the operator*

$$\int_A \int_{K_0} f(h) \Delta_k(h) \pi(h^{\bar{u}}) dh d\bar{u}$$

is summable.

We can choose a homomorphism η of K into C such that $\eta(u^{-1})\pi(u)$ $(u \varepsilon K)$ depends only on $\bar{u}$ (see [5(c), p. 249]). Hence if we put $\bar{\pi}(\bar{u}) = \eta(u^{-1})\pi(u)$ it is clear that $\bar{\pi}$ is a representation of the compact group K_0. Therefore we can find a regular operator B such that $B\bar{\pi}(\bar{u})B^{-1}$ is unitary for every $\bar{u} \varepsilon K_0$. Hence, in view of our definition of the summa-

bility of an operator, it is clear that, without any loss of generality, we may assume that $\bar{\pi}$ itself is a unitary representation. Ω being the set of all equivalence classes of irreducible finite-dimensional representations of K, we denote, as usual, by $\mathfrak{H}_{\mathfrak{D}}$ ($\mathfrak{D} \in \Omega$) the subspace of $\mathfrak{H}$ consisting of those elements which transform under $\pi(K)$ according to $\mathfrak{D}$. Then there exists an integer N such that $\dim \mathfrak{H}_{\mathfrak{D}} \leqq N(d(\mathfrak{D}))^2$ ($\mathfrak{D} \in \Omega$) where $d(\mathfrak{D})$ is the degree of any representation in $\mathfrak{D}$ (see [5(c), Theorem 4]). Since $\bar{\pi}$ is unitary, the subspaces $\mathfrak{H}_{\mathfrak{D}}$ are mutually orthogonal and so we can choose a complete orthonormal set $(\psi_j)_{j \in J}$ in $\mathfrak{H}$ such that each ψ_j lies in some $\mathfrak{H}_{\mathfrak{D}}$ (see [5(a), Theorem 4]). Let $J(\mathfrak{D})$ denote the set of all $j \in J$ for which $\psi_j \in \mathfrak{H}_{\mathfrak{D}}$. Then $(\psi_j)_{j \in J(\mathfrak{D})}$ is an orthonormal base for $\mathfrak{H}_{\mathfrak{D}}$. Put $n(\mathfrak{D}) = (d(\mathfrak{D}))^{-1} \dim \mathfrak{H}_{\mathfrak{D}}$. Then $n(\mathfrak{D})$ is a nonnegative integer. Also if $E_{\mathfrak{D}}$ is the orthogonal projection of $\mathfrak{H}$ on $\mathfrak{H}_{\mathfrak{D}}$, it follows from the Schur orthogonality relations on the compact group K_0 that

$$E_{\mathfrak{D}}\left(\int_{K_0} \pi(h^{\bar{u}})\,d\bar{u}\right)E_{\mathfrak{D}} = \eta(h)E_{\mathfrak{D}}\left(\int_{K_0} \bar{\pi}(\bar{u}\bar{h}\bar{u}^{-1})\,d\bar{u}\right)E_{\mathfrak{D}}$$
$$= d(\mathfrak{D})^{-1}\,sp(E_{\mathfrak{D}}\pi(h)E_{\mathfrak{D}})E_{\mathfrak{D}} = d(\mathfrak{D})^{-1}\zeta_{\mathfrak{D}}(h)E_{\mathfrak{D}} \qquad (h \in A)$$

where $\zeta_{\mathfrak{D}}$ is the character (on K) of the class $\mathfrak{D}$. This shows that if $i, j \in J(\mathfrak{D})$,

$$(\psi_i, Q_f\psi_j) = d(\mathfrak{D})^{-1}\delta_{ij}\int_A f(h)\Delta_k(h)\zeta_{\mathfrak{D}}(h)\,dh$$

where δ_{ij} is the Kronecker symbol and Q_f is the operator of our lemma. Therefore if Ω_π is the set of all $\mathfrak{D} \in \Omega$ such that $\mathfrak{H}_{\mathfrak{D}} \neq \{0\}$,

$$\sum_{i,j \in J} |(\psi_i, Q_f\psi_j)| \leqq N \sum_{\mathfrak{D} \in \Omega_\pi} d(\mathfrak{D}) \left| \int_A f(h)\Delta_k(h)\zeta_{\mathfrak{D}}(h)\,dh \right|.$$

Let $\mathfrak{D}$ be a class in Ω_π and σ a representation in $\mathfrak{D}$. We denote the corresponding representation of $\mathfrak{k}$ also by σ. Since[3] $\mathfrak{k} = \mathfrak{k}' + \mathfrak{c}$, it follows from Schur's lemma that the representation space V of σ is irreducible under $\sigma(\mathfrak{k}')$. Let σ' denote the corresponding representation of $\mathfrak{k}'$ on V. Since $\mathfrak{h}$ is the direct sum of $\mathfrak{c}$ and $\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{k}'$ we may identify linear functions on $\mathfrak{h}'$ with those linear functions on $\mathfrak{h}$ which vanish identically on $\mathfrak{c}$. Now $\mathfrak{k}'$ is semisimple and $\mathfrak{h}'$ is a Cartan subalgebra of $\mathfrak{k}'$ and it is clear that the roots of $\mathfrak{k}'$ with respect to $\mathfrak{h}'$ then coincide with the compact roots of $\mathfrak{g}$. Let $\lambda_{\mathfrak{D}}$ denote the highest weight of σ' (if we take P_k as the set of positive roots of $\mathfrak{k}'$). Then if $2\rho_k = \sum_{\alpha \in P_k} \alpha$ we know (see Weyl [9]) that

$$\Delta_k(\exp H)\zeta_{\mathfrak{D}}(\exp H) = \eta(\exp H) \sum_{s \in \mathfrak{w}_k} \epsilon(s) e^{s\lambda'_{\mathfrak{D}}(H)} \qquad (H \in \mathfrak{h}' \cap \mathfrak{k}_0)$$

where $\lambda'_{\mathfrak{D}} = \lambda_{\mathfrak{D}} + \rho_k$ and $\mathfrak{w}_k$ is the group generated by the Weyl reflexions s_α corresponding to $\alpha \varepsilon P_k$, and $\epsilon(s) = (-1)^r$ where r is the number of negative roots among $s\alpha$ $(\alpha \varepsilon P_k)$. On the other hand the degrees of σ and σ' are the same and therefore (Weyl [9])

$$d(\mathfrak{D}) = \prod_{\alpha \varepsilon P_k} \lambda'_{\mathfrak{D}}(H_\alpha)/\rho_k(H_\alpha).$$

Now if we regard η as a representation of K of degree 1, it is obvious that $\eta(\exp X) = e^{\mu(X)}$ $(X \varepsilon \mathfrak{k}_0)$ where μ is a linear function on $\mathfrak{k}$ which vanishes identically on $\mathfrak{k}'$. Select a base $\Gamma_1, \cdots, \Gamma_r$ for $\mathfrak{c}_0 = \mathfrak{c} \cap \mathfrak{k}_0$ (over R) such that $\exp(t_1\Gamma_1 + \cdots + t_r\Gamma_r) = 1$ in K_0 $(t_1, \cdots, t_r \varepsilon R)$ if and only if $t_1, \cdots, t_r$ are all integers (see [5(c), p. 239]). Then since $\eta(h^{-1})\sigma(h)$ $(h \varepsilon A)$ depends only on $\bar{h}$, it follows that $\sigma(\Gamma_i) = \mu(\Gamma_i) + 2\pi(-1)^{\frac{1}{2}}m_i$ $1 \leqq i \leqq r$ where m_i are integers. We denote by $m_{\mathfrak{D}}$ the linear function on $\mathfrak{h}$ given by $m_{\mathfrak{D}}(H) = 0$ $(H \varepsilon \mathfrak{h}')$ and $m_{\mathfrak{D}}(\Gamma_i) = 2\pi(-1)^{\frac{1}{2}}m_i$ $1 \leqq i \leqq r$. Also put

$$|m_{\mathfrak{D}}| = (m_1^2 + \cdots + m_r^2 + 1)^{\frac{1}{2}}.$$

Then if $\nu_{\mathfrak{D}} = \lambda_{\mathfrak{D}} + m_{\mathfrak{D}} + \rho_k$,

$$\Delta_k(\exp H)\zeta_{\mathfrak{D}}(\exp H) = \eta(\exp H) \sum_{s \varepsilon \mathfrak{w}_k} \epsilon(s) e^{s\nu_{\mathfrak{D}}(H)} \qquad (H \varepsilon \mathfrak{h}_0)$$

and

$$d(\mathfrak{D}) = \prod_{\alpha \varepsilon P_k} \nu_{\mathfrak{D}}(H_\alpha)/\rho_k(H_\alpha)$$

since $sm_{\mathfrak{D}} = m_{\mathfrak{D}}$ $(s \varepsilon \mathfrak{w}_k)$ and $H_\alpha \varepsilon \mathfrak{h}'$ $(\alpha \varepsilon P_k)$.

Let $\mathfrak{U}$ be the subalgebra of $\mathfrak{B}$ generated by $(1, \mathfrak{h})$. We regard elements of $\mathfrak{U}$ as differential operators on A in the usual way so that

$$(Hg)(h) = \{(d/dt)g(h \exp tH)\}_{t=0} \qquad (h \varepsilon A, H \varepsilon \mathfrak{h}_0, t \varepsilon R, g \varepsilon C_c^\infty(A)).$$

Since any element $s \varepsilon \mathfrak{w}_k$ permutes the compact roots among themselves, it follows that

$$\prod_{\alpha \varepsilon P_k} s\nu_{\mathfrak{D}}(H_\alpha) = \pm \prod_{\alpha \varepsilon P_k} \nu_{\mathfrak{D}}(H_\alpha)$$

and therefore it is obvious that there exists an element $z_0 \varepsilon \mathfrak{U}$ such that

$$z_0(e^{s\nu_{\mathfrak{D}}}) = d(\mathfrak{D})^2 |m_{\mathfrak{D}}|^2 e^{s\nu_{\mathfrak{D}}} \qquad (s \varepsilon \mathfrak{w}_k).$$

Here $e^{s\nu_{\mathfrak{D}}}$ is the function g on A given by[12] $g(\exp H) = e^{s\nu_{\mathfrak{D}}(H)}$ $(H \varepsilon \mathfrak{h}_0)$. Hence if $g_{\mathfrak{D}}(h) = \eta(h^{-1})\Delta_k(h)\zeta_{\mathfrak{D}}(h)$ it follows that $z_0 g_{\mathfrak{D}} = d(\mathfrak{D})^2 |m_{\mathfrak{D}}|^2 g_{\mathfrak{D}}$. Now let ϕ denote the automorphism of $\mathfrak{U}$ over C such that $\phi(H) = -H$

[12] It follows easily from the discussion at the beginning of § 5 that such a function on A actually exists.

and $\phi(1)=1$ $(H \varepsilon \mathfrak{h})$. Then if $z=\phi(z_0)$, $f'(h)=f(h)\eta(h)$ $(h \varepsilon A)$ and m is a positive integer, it is clear that

$$d(\mathfrak{D})^{2m} | m_{\mathfrak{D}} |^{2m} \int_A f'(h) g_{\mathfrak{D}}(h) dh$$
$$= \int_A f'(h) (z_0{}^m g_{\mathfrak{D}})(h) dh = \int_A (z^m f')(h) g_{\mathfrak{D}}(h) dh.$$

But $| e^{s\nu_{\mathfrak{D}}(H)} | = 1$ $(H \varepsilon \mathfrak{h}_0)$, hence $| g_{\mathfrak{D}}(h) | \leqq w$ $(h \varepsilon A)$ where w is the order of $\mathfrak{w}_k$. Hence

$$| \int_A (z^m f')(h) g_{\mathfrak{D}}(h) dh | \leqq w \int_A | (z^m f')(h) | dh$$

and therefore

$$\sum_{\mathfrak{D} \varepsilon \Omega_\pi} d(\mathfrak{D}) | \int_A f(h) \Delta_k(h) \zeta_{\mathfrak{D}}(h) dh |$$
$$\leqq \sum_{\mathfrak{D} \varepsilon \Omega_\pi} d(\mathfrak{D})^{1-2m} | m_{\mathfrak{D}} |^{-2m} w \int_A | z^m f'(h) | dh.$$

However if m is sufficiently large we know (see [5(c), p. 240]) that

$$\sum_{\mathfrak{D} \varepsilon \Omega_\pi} d(\mathfrak{D})^{1-2m} | m_{\mathfrak{D}} |^{-2m} < \infty$$

and this proves that Q_f is summable.

Now every summable operator has a trace [5(c), Lemma 1]. Let $\tau_\pi(f)$ denote the trace of Q_f. Then the above proof shows that it is possible to choose a positive integer m and a real constant M such that

$$| \tau_\pi(f) | \leqq M \int_A | z^m f'(h) | dh$$

for all $f \varepsilon C_c{}^\infty(A)$. (Here $f' = f\eta$). This shows that τ_π is a distribution (see Schwartz [8]) on A of finite order. Moreover

$$\tau_\pi(f) = \sum_{\mathfrak{D} \varepsilon \Omega_\pi} \sum_{j \varepsilon J(\mathfrak{D})} (\psi_j, Q_f \psi_j) = \sum_{\mathfrak{D} \varepsilon \Omega_\pi} n(\mathfrak{D}) \int_A f(h) \Delta_k(h) \zeta_{\mathfrak{D}}(h) dh$$

and this shows that τ_π does not change if we replace π by another infinitesimally equivalent representation.

Now let Λ_0 be a real linear function on $\mathfrak{h}$ satisfying the following three conditions (cf. Theorem 3 of [5(f)]):

(1) $\Lambda_0(H_\alpha)$ is a nonnegative integer for every $\alpha \varepsilon P_k$.

(2) $\Lambda_0(H_\beta) = 0$ for every noncompact positive root which is not totally positive.

(3) $\Lambda_0(H_\gamma)+\rho(H_\gamma)\leqq 0$ for every totally positive root γ. (Here 2ρ is the sum of all positive roots).

We consider the quasi-simple irreducible representation π of G defined in Theorem 2 corresponding to Λ_0. Put $\tau_{\Lambda_0}=\tau_\pi$ and let $\mathfrak{H}$ denote the representation space of π. Our object is to compute τ_{Λ_0}. We shall see in another paper that τ_{Λ_0} is intimately related with T_{Λ_0} (see however [5(g)]).

We keep to the notation of the proof of Lemma 24. By going over, if necessary, to an equivalent representation we may again assume that π is unitary. For any linear function Λ on $\mathfrak{h}$ let $\mathfrak{H}_\Lambda$ denote the set of all elements $\phi\,\varepsilon\,\mathfrak{H}$ such that $\pi(\exp H)\phi=e^{\Lambda(H)}\phi$ $(H\,\varepsilon\,\mathfrak{h}_0)$. Then it is clear that the subspaces $\mathfrak{H}_\Lambda$ are mutually orthogonal. Put $\mathfrak{H}^0=\sum_{\mathfrak{D}\,\varepsilon\,\Omega}\mathfrak{H}_{\mathfrak{D}}$ and $\mathfrak{H}_\Lambda{}^0=\mathfrak{H}_\Lambda\cap\mathfrak{H}^0$. Since $\mathfrak{H}_{\mathfrak{D}}$ is finite-dimensional and fully reducible under $\pi(A)$, it is obvious that $\mathfrak{H}^0=\sum_\Lambda\mathfrak{H}_\Lambda{}^0$. Let E_Λ denote the orthogonal projection of $\mathfrak{H}$ on $\mathfrak{H}_\Lambda$. Since $\mathfrak{H}^0$ is dense in $\mathfrak{H}$, $E_\Lambda\mathfrak{H}^0$ is dense in $\mathfrak{H}_\Lambda$. But it is clear that $E_\Lambda\mathfrak{H}^0\subset\mathfrak{H}^0$ and therefore $E_\Lambda\mathfrak{H}^0\subset\mathfrak{H}_\Lambda{}^0$. However $\dim\mathfrak{H}_\Lambda{}^0<\infty$ (corollary to Lemma 21 of [5(f)]) and so it follows that $\mathfrak{H}_\Lambda=\mathfrak{H}_\Lambda{}^0$.

Let P_+ denote the set of all totally positive roots of $\mathfrak{g}$ and let $\mathfrak{h}_+$ be the open subset of $\mathfrak{h}$ consisting of all those element H for which $|e^{\gamma(H)}|>1$ for every $\gamma\,\varepsilon\,P_+$. Since every root takes pure imaginary values on $\mathfrak{h}_0$, it is obvious that $\mathfrak{h}_++\mathfrak{h}_0\subset\mathfrak{h}$. Let σ_β $(\beta\,\varepsilon\,P_+)$ denote the hyperplane in the real Euclidean space $\mathfrak{h}^*=(-1)^{\frac{1}{2}}\mathfrak{h}_0$ defined by the equation $\beta(H)=0$. Consider the complement $\mathfrak{h}_1{}^*$ of $\bigcup_{\beta\,\varepsilon\,P_+}\sigma_\beta$ in $\mathfrak{h}^*$. Let $(\alpha_1,\cdots,\alpha_l)$ be a fundamental system of positive roots and H_0 a point in $\mathfrak{h}$ such that $\alpha_i(H_0)=1$ $1\leqq i\leqq l$. Then $H_0\,\varepsilon\,\mathfrak{h}_1{}^*$ and if $\mathfrak{h}_+{}^*$ is the connected component of H_0 in $\mathfrak{h}_1{}^*$, it is obvious that $\mathfrak{h}_+{}^*\subset\mathfrak{h}^*\cap\mathfrak{h}_+$. Conversely $\mathfrak{h}^*\cap\mathfrak{h}_+$ is a convex subset of $\mathfrak{h}_1{}^*$ and therefore it is connected. Hence $\mathfrak{h}_+{}^*=\mathfrak{h}^*\cap\mathfrak{h}_+$ and therefore $\mathfrak{h}_+=\mathfrak{h}_0+\mathfrak{h}_+{}^*$. In particular $tH_0\,\varepsilon\,\mathfrak{h}_+{}^*$ for every $t>0$ and so zero lies in the closure of $\mathfrak{h}_+$. Now consider the complex abelian Lie group $\tilde{A}_c$ defined in Section 3 and let ${}_+\tilde{A}_c$ denote the subset of those $h\,\varepsilon\,\tilde{A}_c$ which can be written in the form $h=\exp H$ $(H\,\varepsilon\,\mathfrak{h}_+)$. It is clear that ${}_+\tilde{A}_c$ is an open connected subset of $\tilde{A}_c$ whose closure contains 1. Also ${}_+\tilde{A}_cA={}_+\tilde{A}_c$. We shall now define a bounded operator $\pi(h)$ on $\mathfrak{H}$ for every $h\,\varepsilon\,{}_+\tilde{A}_c$. Let $\mathfrak{F}_\pi$ be the set of those linear function Λ on $\mathfrak{h}$ for which $\mathfrak{H}_\Lambda\neq\{0\}$. Then if $\exp H=1$ in A $(H\,\varepsilon\,\mathfrak{h}_0)$ it is obvious that $e^{\Lambda(H)}=1$ and therefore, as we have seen in Section 5, there exists a holomorphic character ξ_Λ of $\tilde{A}_c$ such that $\xi_\Lambda(\exp H)=e^{\Lambda(H)}$ $(H\,\varepsilon\,\mathfrak{h})$. Let ψ be the element in $\mathfrak{H}$ corresponding to Theorem 2. Then $\psi\,\varepsilon\,\mathfrak{H}_{\Lambda_0}$ and $\mathfrak{H}^0=\pi(\mathfrak{B})\psi$. Moreover if $V=\pi(\mathfrak{X})\psi$, $\dim V<\infty$ (Lemmas 8 and 10) and

V is irreducible under $\pi(\mathfrak{k})$ (Lemma 2 of [5(f)]). Let $\mathfrak{D}_0$ denote the class[13] (in Ω_π) and $\Lambda_0, \Lambda_1, \cdots, \Lambda_r$ all the (distinct) weights of this representation of $\mathfrak{k}$ on V. Then if $\gamma_1, \cdots, \gamma_q$ are all the (distinct) totally positive roots of $\mathfrak{g}$, every $\Lambda \in \mathfrak{F}_\pi$ can be written in the form

$$\Lambda = \Lambda_i - (m_1\gamma_1 + \cdots + m_q\gamma_q)$$

for some i $(0 \leqq i \leqq r)$ and some non-negative integers $m_1, \cdots, m_q$ (see the corollary to Lemma 21 of [5(f)]). Therefore if $h \in {}_+\tilde{A}_c$

$$|\xi_\Lambda(h)| \leqq \max_{0 \leqq i \leqq r} |\xi_{\Lambda_i}(h)| \qquad (\Lambda \in \mathfrak{F}_\pi).$$

Hence it is clear that the infinite sum $\sum_{\Lambda \in \mathfrak{F}_\pi} \xi_\Lambda(h) E_\Lambda \phi$ converges in $\mathfrak{H}$ for any $h \in {}_+\tilde{A}_c$ and $\phi \in \mathfrak{H}$. Let $\pi(h)\phi$ denote the limit of this sum. Then

$$|\pi(h)\phi|^2 \leqq \max_{0 \leqq i \leqq r} |\xi_{\Lambda_i}(h)|^2 |\phi|^2$$

and therefore $\pi(h)$ is a bounded operator on $\mathfrak{H}$. It is obvious from its definition that if $h_+ \in {}_+\tilde{A}_c$ and $h \in A$, $\pi(hh_+) = \pi(h)\pi(h_+) = \pi(h_+)\pi(h)$.

Let $\zeta_{\mathfrak{D}_0}$ denote the character of the class $\mathfrak{D}_0$. Then if $d_i = \dim(V \cap \mathfrak{H}_{\Lambda_i})$ $0 \leqq i \leqq r$,

$$\zeta_{\mathfrak{D}_0}(h) = \sum_{0 \leqq i \leqq r} d_i \xi_{\Lambda_i}(h)$$

for any $h \in A$ and we can extend $\zeta_{\mathfrak{D}_0}$ to a holomorphic function on $\tilde{A}_c$ by means of the above formula. Let λ be a linear function on $\mathfrak{h}$ such that $\lambda(H_\alpha)$ is an integer for every compact root α. Then as we have seen in Section 5, there exists a holomorphic character ξ_λ of $\tilde{A}_c$, such that $\xi_\lambda(\exp H) = e^{\lambda(H)}$ $(H \in \mathfrak{h})$. Put $\rho_+ = \frac{1}{2}\sum_{\gamma \in P_+} \gamma$, $\rho_k = \frac{1}{2}\sum_{\alpha \in P_k} \alpha$ and $\rho_0 = \rho_+ + \rho_k$. Then if s_α is the Weyl reflexion corresponding to a compact root α, it follows from Lemma 10 of [5(f)] that $s_\alpha\rho_+ = \rho_+$ and therefore $\rho_+(H_\alpha) = 0$. Also if β is any root, $\beta(H_\alpha)$ and $\rho_k(H_\alpha)$ are integers (see Weyl [9]). This shows that we can construct the corresponding characters $\xi_{\rho_+}, \xi_{\rho_k}, \xi_{\rho_0}, \xi_\beta$. Put

$$\Delta_k(h) = \xi_{\rho_k}(h) \prod_{\alpha \in P_k} \{1 - \xi_\alpha(h^{-1})\} = \prod_{\alpha \in P_k} (e^{\frac{1}{2}\alpha(H)} - e^{-\frac{1}{2}\alpha(H)}),$$

$$\Delta_+(h) = \xi_{\rho_+}(h) \prod_{\gamma \in P_+} \{1 - \xi_\gamma(h^{-1})\} = \prod_{\gamma \in P_+} \{e^{\frac{1}{2}\gamma(H)} - e^{-\frac{1}{2}\gamma(H)}\}$$

where $h = \exp H \in \tilde{A}_c$ $(H \in \mathfrak{h})$.

LEMMA 25. *If $h \in {}_+\tilde{A}_c$ the operator $\pi(h)$ is summable and*

$$sp\,\pi(h) = \{\Delta_+(h)\}^{-1} \xi_{\rho_+}(h) \zeta_{\mathfrak{D}_0}(h).$$

[13] As usual we identify finite-dimensional irreducible representations of $\mathfrak{k}$ with those of K.

Moreover the series $\sum_{\Lambda \varepsilon \mathfrak{F}_\pi} (\dim \mathfrak{H}_\Lambda) | \xi_\Lambda(h) |$ *converges uniformly on every compact subset of* ${}_+\tilde{A}_c$.

Let $h = \exp H$ $(H \varepsilon \mathfrak{h}_+)$. Then from the corollary to Lemma 21 of [5(f)],

$$\sum_{\Lambda \varepsilon \mathfrak{F}_\pi} (\dim \mathfrak{H}_\Lambda) | \xi_\Lambda(h) |$$
$$= \sum_{0 \leqq i \leqq r} d_i \sum_{m_1, \cdots, m_q \geqq 0} | \exp\{\Lambda_i(H) - m_1\gamma_1(H) - \cdot\cdot\cdot - m_q\gamma_q(H)\} |.$$

Since the real part of $\gamma_j(H)$ is positive, the series converges and clearly its convergence is uniform if H remains in a compact subset of $\mathfrak{h}_+$. Moreover since $\pi(h)$ coincides with $\xi_\Lambda(h)E_\Lambda$ on $\mathfrak{H}_\Lambda$, it is now clear that $\pi(h)$ is summable and

$$sp\pi(h) = \sum_{\Lambda \varepsilon \mathfrak{F}_\pi} (\dim \mathfrak{H}_\Lambda)\xi_\Lambda(h)$$
$$= \sum_{0 \leqq i \leqq r} d_i \sum_{m_1, \cdots, m_q \geqq 0} \exp(\Lambda_i(H) - m_1\gamma_1(H) - \cdot\cdot\cdot - m_q\gamma_q(H))$$
$$= \prod_{1 \leqq j \leqq q} \{1 - e^{-\gamma_j(H)}\}^{-1}\zeta_{\mathfrak{D}_0}(\exp H) = \{\Delta_+(h)\}^{-1}\xi_{\rho_+}(h)\zeta_{\mathfrak{D}_0}(h).$$

For any $f \varepsilon C_c^\infty(A)$ and $h_+ \varepsilon {}_+\tilde{A}_c$, put

$$Q_f(h_+) = \int_A f(h)\Delta_k(hh_+)\pi(hh_+)\,dh, \quad Q_f = \int_A \int_{K_0} f(h)\Delta_k(h)\pi(h^{\bar{u}})\,dhd\bar{u}.$$

LEMMA 26. *If* $f \varepsilon C_c^\infty(A)$ *the operator* $Q_f(h_+)$ *is summable and*

$$\lim_{h_+ \to 1} spQ_f(h_+) = spQ_f \qquad (h_+ \varepsilon {}_+\tilde{A}_c).$$

Let $\Lambda \varepsilon \mathfrak{F}_\pi$. Then it is obvious that

$$Q_f(h_+)E_\Lambda = \{\int_A f(h)\Delta_k(hh_+)\xi_\Lambda(hh_+)\,dh\}E_\Lambda,$$

and therefore, in order to show that $Q_f(h_+)$ is summable, it is enough to prove that

$$\sum_{\Lambda \varepsilon \mathfrak{F}_\pi} (\dim \mathfrak{H}_\Lambda) | \int_A f(h)\Delta_k(hh_+)\xi_\Lambda(hh_+)\,dh | < \infty.$$

But this is obvious in view of the fact that

$$\sum_{\Lambda \varepsilon \mathfrak{F}_\pi} (\dim \mathfrak{H}_\Lambda) | \xi_\Lambda(hh_+) |$$

converges uniformly (with respect to h) on every compact subset of A (Lemma 25).

Now we use the notation of the proof of Lemma 24. Since $Q_f(h_+)$ is summable,

$$spQ_f(h_+) = \sum_{\mathfrak{D}\varepsilon\Omega_\pi} sp(E_{\mathfrak{D}}Q_f(h_+)E_{\mathfrak{D}}).$$

Then if we define η, μ, $\nu_{\mathfrak{D}}$, $n(\mathfrak{D})$ and $m(\mathfrak{D})$ $(\mathfrak{D}\varepsilon\Omega_\pi)$ as before,

$$\eta(\exp H) = e^{\mu(H)}, \Delta_k(\exp H)\zeta_{\mathfrak{D}}(\exp H) = e^{\mu(H)} \sum_{s\varepsilon\mathfrak{w}_k} \epsilon(s) e^{s\nu_{\mathfrak{D}}(H)} \qquad (H\varepsilon\mathfrak{h}_0)$$

and it is clear that we can extend η and $\zeta_{\mathfrak{D}}$ to holomorphic functions on $\tilde{A}_c$ so that the above relations actually hold for all $H\varepsilon\mathfrak{h}$. Then if $\mathfrak{D}\varepsilon\Omega_\pi$,

$$sp(E_{\mathfrak{D}}Q_f(h_+)E_{\mathfrak{D}}) = n(\mathfrak{D}) \int_A f(h)\Delta_k(hh_+)\zeta_{\mathfrak{D}}(hh_+)dh$$

and therefore

$$spQ_f(h_+) = \sum_{\mathfrak{D}\varepsilon\Omega_\pi} n(\mathfrak{D}) \int_A f(h)\Delta_k(hh_+)\zeta_{\mathfrak{D}}(hh_+)dh.$$

Now put $\nu'_{\mathfrak{D}} = \nu_{\mathfrak{D}} + \mu_0$ where μ_0 is the restriction of μ on $\mathfrak{h}$,

$$g_{\mathfrak{D}}(h) = \eta(h^{-1})\Delta_k(h)\zeta_{\mathfrak{D}}(h) = \sum_{s\varepsilon\mathfrak{w}_k} \epsilon(s)\xi_{s\nu_{\mathfrak{D}}}(h) \qquad (h\varepsilon\tilde{A}_c)$$

and $f'(h) = f(h)\eta(h)$ $(h\varepsilon A)$. It is obvious from the definition of $\nu'_{\mathfrak{D}}$ that $\nu'_{\mathfrak{D}} - \rho_k \varepsilon \mathfrak{F}_\pi$ and therefore $s(\nu'_{\mathfrak{D}} - \rho_k)\varepsilon\mathfrak{F}_\pi$ for $s\varepsilon\mathfrak{w}_k$ (corollary to Lemma 6 of [5(f)]). Hence

$$|\xi_{s\nu_{\mathfrak{D}}}(h)| \leqq \max_{\substack{0\leqq i\leqq r\\ s\varepsilon\mathfrak{w}_k}} |\xi_{\Lambda_i+s\rho_k}(h)|\ |\eta(h^{-1})| \qquad (h\varepsilon{}_+\tilde{A}_c).$$

We can now argue as in the proof of Lemma 24 and show that the series

$$\sum_{\mathfrak{D}\varepsilon\Omega_\pi} n(\mathfrak{D}) \left| \int_A f(h)\Delta_k(hh_+)\zeta_{\mathfrak{D}}(hh_+)dh \right|$$

converges uniformly with respect to h_+ on $B\cap{}_+\tilde{A}_c$ where B is a compact neighbourhood of 1 in $\tilde{A}_c$. Hence it follows that

$$\begin{aligned}\operatorname*{Lim}_{h_+\to 1} spQ_f(h_+) &= \sum_{\mathfrak{D}\varepsilon\Omega_\pi} n(\mathfrak{D}) \operatorname*{Lim}_{h_+\to 1} \int_A f(h)\Delta_k(hh_+)\zeta_{\mathfrak{D}}(hh_+)dh \\ &= \sum_{\mathfrak{D}\varepsilon\Omega_\pi} n(\mathfrak{D}) \int_A f(h)\Delta_k(h)\zeta_{\mathfrak{D}}(h)dh = spQ_f.\end{aligned}$$

Now put

$$\Xi_{\Lambda_0} = \xi_{\rho_+}\zeta_{\mathfrak{D}_0}\Delta_k = \sum_{s\varepsilon\mathfrak{w}_k} \epsilon(s)\xi_{s(\Lambda_0+\rho_0)}.$$

COROLLARY 1. $\tau_{\Lambda_0}(f) = \lim_{h_+\to 1} \int_A f(h)(\Xi_{\Lambda_0}(hh_+)/\Delta_+(hh_+))dh \quad (h_+ \varepsilon {}_+\tilde{A}_c)$ *for* $f \varepsilon C_c^\infty(A)$.

This is an immediate consequence of Lemma 26 and the fact that $\tau_{\Lambda_0}(f) = spQ_f$.

COROLLARY[14] 2. $\Delta_+\tau_{\Lambda_0} = \Xi_{\Lambda_0}$ *on* A.

Let H_+ be an element in $\mathfrak{h}_+{}^*$ so that $\gamma(H_+)$ is real and positive for every $\gamma \varepsilon P_+$. Then if $h_+(t) = \exp tH_+$ $(t > 0)$ we know from the above lemma that

$$\tau_{\Lambda_0}(\Delta_+ f) = \lim_{t\to 0} \int_A \Delta_+(h) f(h)\{\Xi_{\Lambda_0}(hh_+(t))/\Delta_+(hh_+(t))\}dh$$

for $f \varepsilon C_c^\infty(A)$. Now we claim that there exists a constant M such that

$$|\Delta_+(h)\Xi_{\Lambda_0}(hh_+(t))/\Delta_+(hh_+(t))| \leqq M$$

for all $h \varepsilon A$ provided t is sufficiently small and positive. This is seen as follows. If θ and ϵ are real numbers and $\epsilon \geqq \frac{1}{2}$,

$$\begin{aligned} |(e^{(-1)^{\frac{1}{2}}\theta} - 1)/(\epsilon e^{(-1)^{\frac{1}{2}}\theta} - 1)|^2 &= (2 - 2\cos\theta)/(1 + \epsilon^2 - 2\epsilon\cos\theta) \\ &= 2(1-\cos\theta)/\{(1-\epsilon)^2 + 2\epsilon(1-\cos\theta)\} \leqq 1/\epsilon \leqq 2. \end{aligned}$$

Hence if q is the number of totally positive roots of $\mathfrak{g}$, it is clear that

$$|\Delta_+(h)/\Delta_+(hh_+(t))| \leqq 2^q$$

provided t is sufficiently small and positive. Our assertion is now obvious. Therefore by Lebesgue's Theorem

$$\tau_{\Lambda_0}(\Delta_+ f) = \int_A f(h) \lim_{t\to 0} \{\Delta_+(h)\Xi_{\Lambda_0}(hh_+(t))/\Delta_+(hh_+(t))\}dh.$$

But

$$\lim_{t\to 0} \Delta_+(h)\Xi_{\Lambda_0}(hh_+(t))/\Delta_+(hh_+(t)) = \begin{cases} 0 \text{ if } \Delta_+(h) = 0 \\ \Xi_{\Lambda_0}(h) \text{ if } \Delta_+(h) \neq 0 \end{cases} \quad (h \varepsilon A)$$

and since the set of all $h \varepsilon A$ for which $\Delta_+(h) = 0$ is of Haar measure zero, we get

$$\tau_{\Lambda_0}(\Delta_+ f) = \int_A f(h)\Xi_{\Lambda_0}(h)dh.$$

This proves the corollary.

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[14] Here we use the standard terminology of the theory of distributions (Schwartz [8]).

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SPECTRAL ISOMORPHISMS FOR SOME RINGS OF INFINITE MATRICES ON A BANACH SPACE.*

By G. L. Krabbe.

1. Introduction. Suppose $p > 1$, and let l_p denote the set of all sequences c such that $\sum_{n=-\infty}^{\infty} | c_n |^p < \infty$. If a is a sequence $\{a_n\}$, the matrix $(a_{n-\nu})$ is usually known as a *Laurent matrix* when $n, \nu = 0, \pm 1, \pm 2, \cdots$; it represents the transformation $\langle a \rangle_p$ which maps any member c of l_p on a sequence b defined by

$$b_n = \sum_{\nu=-\infty}^{\infty} a_{n-\nu} c_\nu \qquad (n = 0, \pm 1, \pm 2, \cdots).$$

If F belongs to the ring (L^∞) of essentially bounded summable functions on $[-\pi, \pi]$, then ΛF will denote the sequence of Fourier coefficients of F. Restricting themselves to the case $p = 2$, O. Toeplitz [14, pp. 499-502] and F. Riesz [12] have shown that, when $\mathfrak{F} = (L^\infty)$, then

(i) *the mapping $F \to \langle \Lambda F \rangle_p$ is a continuous isomorphism of $\mathfrak{F}$ into the ring $\mathfrak{E}$ of bounded operators on l_p.*

When $\mathfrak{F}$ is the ring (C) of continuous functions (and again with the restriction $p = 2$), they established (see [4]) that

(ii) *the spectrum of $\langle \Lambda F \rangle_p$ is the image $F([-\pi, \pi])$, when $F \varepsilon \mathfrak{F}$,*

(iii) *if $F \varepsilon \mathfrak{F}$, then the inverse Q of the operator $\langle \Lambda F \rangle_p$ exists if and only if F does not vanish on $[-\pi, \pi]$. When Q exists, then $Q = \langle \Lambda G \rangle_p$, where $G(\theta) = [F(\theta)]^{-1}$.*

Assume henceforth $p > 1$; we will show in 4.2 that (i) holds when $\mathfrak{F}$ is the ring (BV) of functions of bounded variation on $[-\pi, \pi]$. The above results of Riesz and Toeplitz prompted us to consider (in [6]) the extent to which (ii) holds when F is in the ring (CBV) of continuous functions in (BV); it was found that the spectrum of $\langle \Lambda F \rangle_p$ is then a connected subset of $F([-\pi, \pi])$. In the present paper, we prove that (ii) and (iii) hold

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when $\mathfrak{J}$ is the ring (AC) of absolutely continuous functions. From these results follows that, if $F\varepsilon(AC)$ and $T=\langle\Lambda F\rangle_p$, then

$$\sup\{|F(\theta)|:|\theta|\leqq\pi\}\leqq\|T\|.$$

The equality sign holds when $p=2$ (see 4.4 and Riesz [12]). In the general case $p>1$, we show that T belongs to the ring $\mathfrak{E}[H]$ generated by the operator H which is represented by the Laurent matrix $(\alpha_{n\nu})$, where $\alpha_{n\nu}=i(-1)^{n+\nu}/(n-\nu)$ and $\alpha_{nn}=0$. Moreover, the mapping $F\rightarrow\langle\Lambda F\rangle_p$ is the only continuous homomorphism of (AC) into $\mathfrak{E}$ which maps on H the identity-function.

1. 1. Application. Suppose f belongs to the ring $\mathfrak{U}$ of all functions f such that $f(\lambda)=\sum a_n\lambda^n$ $(n=-\infty\cdots\infty)$ for some a in l_1. Note that the perceding series is the Laurent-expansion of f in some annulus containing $\Gamma_1=\{\lambda:|\lambda|=1\}$; two members of $\mathfrak{U}$ will be considered equal when they coincide on Γ_1. Set $\Delta f=\Lambda F$, where $F(\theta)=f(e^{i\theta})$; since $F(\theta)=\sum a_n e^{in\theta}$, we have $a=\Delta f$. The Laurent matrix $(a_{n-\nu})$ again represents an operator $\langle\Delta f\rangle_p$. Toeplitz ([15], [16]) has proved that, when $p=2$ and $\mathfrak{J}=\mathfrak{U}$, then

(i′) *the mapping* $f\rightarrow\langle\Delta f\rangle_p$ *is an isomorphism of* $\mathfrak{J}$ *into the ring* $\mathfrak{E}$,

(ii′) *the spectrum of* $\langle\Delta f\rangle_p$ *is the image of* Γ_1 *by* f, *when* $f\varepsilon\mathfrak{J}$,

(iii′) *if* $f\varepsilon\mathfrak{J}$, *then the inverse* Q *of the operator* $\langle\Delta f\rangle_p$ *exists if and only if* f *does not vanish on* Γ_1. *When* Q *exists, then* $Q=\langle\Delta g\rangle_p$, *where* $g(\lambda)=[f(\lambda)]^{-1}$.

It follows readily from our results that (i′)-(iii′) hold for any $p>1$, when we take for $\mathfrak{J}$ the larger class of all functions f such that $f(e^{i\theta})$ is an absolutely continuous function of θ $(|\theta|\leqq\pi)$.

2. Spectral mappings. Suppose $\mathfrak{R}$ is a Banach algebra with unit $\mathbf{1}$; we write $R=\mathfrak{R}\,\mathrm{lm}\,R_n$ if $\lim\|R-R_n\|=0$ $(n\rightarrow\infty$, and in the norm of $\mathfrak{R})$. Let $\mathfrak{R}'$ be the set of all T in $\mathfrak{R}$ such that T has an inverse (denoted T^{-1}) in $\mathfrak{R}$. The spectrum $\sigma(R)$ of some R in $\mathfrak{R}$ is the set of all complex λ such that $\lambda\mathbf{1}-R\not\varepsilon\mathfrak{R}'$.

Suppose $\mathfrak{S}$ is a fixed compact subset of the plane. The Banach algebra $\boldsymbol{C}(\mathfrak{S})$ of all complex-valued functions on $\mathfrak{S}$ has the norm

$$\|A\|_\infty=\sup\{|A(\theta)|:\theta\varepsilon\mathfrak{S}\};$$

the product $A\cdot B$ of two members of $\boldsymbol{C}(\mathfrak{S})$ is the function F defined by $F(\theta)=A(\theta)\cdot B(\theta)$.

2.1 *Remark.* Denoting by I the member of $\boldsymbol{C}(\mathfrak{s})$ defined by $I(\theta)=\theta$, we have $I^n(\theta)=\theta^n$ and $I^0(\theta)=1$; note that I^0 is the unit-element of $\boldsymbol{C}(\mathfrak{s})$. It is easily verified that $\sigma(A)$ is the image $A(\mathfrak{s})$ of $\mathfrak{s}$ by A (when $A\,\varepsilon\,\boldsymbol{C}(\mathfrak{s})$).

2.2 Definition. *Assume that $\mathfrak{T}$ is a subset of a given Banach algebra. If ϕ is a mapping of $\mathfrak{T}$ into some Banach algebra, we say that ϕ is "spectral" if $\sigma(T)=\sigma(\phi(T))$ for all T in $\mathfrak{T}$.*

2.3 Lemma. *If $\mathfrak{T}$ is a subset of some Banach algebra, and if ϕ is a spectral mapping of $\mathfrak{T}$ into $\boldsymbol{C}(\mathfrak{s})$, then $\|\phi(T)\|\leqq\|T\|$ when $T\,\varepsilon\,\mathfrak{T}$.*

Proof. Set $A=\phi(T)$. It is easily seen that $\|A\|_\infty=\sup\{|\lambda|:\lambda\,\varepsilon\,\sigma(A)\}$, and therefore $\|A\|_\infty=\sup\{|\lambda|:\lambda\,\varepsilon\,\sigma(T)\}\leqq\lim\|T^n\|^{1/n}\leqq\|T\|$; the first inequality always holds for a member T of some Banach algebra ([7], 24 A).

2.4 Lemma. *Let $\mathfrak{F}$ denote either $\boldsymbol{C}(\mathfrak{s})$ or one of its subsets (AC) and (CBV). Suppose Φ is a spectral homomorphism of some Banach algebra $\mathfrak{X}$ into $\boldsymbol{C}(s)$. If $T\,\varepsilon\,\mathfrak{X}$ and $A=\Phi(T)\,\varepsilon\,\mathfrak{F}$, then*

(iv) $\|A\|_\infty\leqq\|T\|$; $\sigma(T)=A(\mathfrak{s})$; $T^{-1}\varepsilon\,\mathfrak{X}\leftrightarrows 0\notin A(\mathfrak{s})$,

(v) *if $T^{-1}\varepsilon\,\mathfrak{X}$ then $A^{-1}=\Phi(T^{-1})\,\varepsilon\,\mathfrak{F}$.*

Remark. Since $\sigma(A)=A(\mathfrak{s})$ (see 2.1), we have $0\notin A(\mathfrak{s})\leftrightarrows A(\theta)\neq 0$ for all θ in $\mathfrak{s}\leftrightarrows 0\notin\sigma(A)$; note that $A^{-1}(\theta)=[A(\theta)]^{-1}$ when $0\notin A(\mathfrak{s})$.

Proof. The first two statements of (iv) follow from 2.3, 2.2, and 2.1. On the other hand, $T^{-1}\varepsilon\,\mathfrak{X}\leftrightarrows 0\notin\sigma(T)=\sigma(A)\leftrightarrows A^{-1}\varepsilon\,\mathfrak{F}$; the proof is now concluded by noting that $A^{-1}\varepsilon\,\mathfrak{F}\leftrightarrows 0\notin A(\mathfrak{s})$ in each of the cases considered.

2.5 *Remark.* If $\mathfrak{s}$ is connected and if $\mathfrak{X}$ is abelian, then the existence of a spectral mapping of $\mathfrak{X}$ into $\boldsymbol{C}(\mathfrak{s})$ implies that $\mathfrak{X}$ is irreducible (in the sense that $\mathbf{1}$ and $\mathbf{O}$ are the only members X of $\mathfrak{X}$ such that $X^2=X$). This is because a necessary and sufficient condition for irreducibility is that the spectra of all members be connected sets ([5], p. 454).

2.6 *Remark.* The inverse V of the mapping $F\to\Lambda F$ is a spectral isomorphism of the Banach algebra l_1 into $\boldsymbol{C}(\mathfrak{s})$ (where $\mathfrak{s}=[-\pi,\pi]$; see 4.5). A related example of a spectral isomorphism is also found in 4.5. Note that (iv)-(v) include the statements (ii)-(iii) when $T=\langle\Lambda A\rangle_p$.

2.7 Theorem. *Suppose $\mathfrak{T}$ is dense in an abelian Banach algebra $\mathfrak{X}$. If ϕ is a spectral homomorphism of $\mathfrak{T}$ into $\boldsymbol{C}(\mathfrak{s})$, then ϕ can be extended to a spectral homomorphism Φ of $\mathfrak{X}$ into $\boldsymbol{C}(\mathfrak{s})$.*

Proof. The following property will be needed. If R and R_n are members of some abelian Banach algebra $\mathfrak{R}$, then (assuming $n \to \infty$ throughout)

(a) $$R = \mathfrak{R} \lim R_n \quad \text{implies} \quad \sigma(R) = \lim \sigma(R_n).$$

In this statement (proved in [11]), lim is the Hausdorff limit. From the continuity of ϕ (see 2.3) follows (using a well known theorem; cf. [7], 7 F) that ϕ can be extended to a continuous mapping Φ on $\mathfrak{X}$;

(1) $$\Phi(T) = \phi(T) \text{ for all } T \text{ in } \mathfrak{T}.$$

The proof is completed by establishing the spectrality of Φ. For any X in $\mathfrak{X}$ there exists some T_n in $\mathfrak{T}$ such that $X = \mathfrak{X} \lim T_n$; accordingly, if Ψ is a continuous mapping on $\mathfrak{X}$, then (by (a)),

(2) $$\sigma(\Psi(X)) = \lim \sigma(\Psi(T_n)) \qquad (n \to \infty).$$

This holds therefore for $\Psi = \Phi$ and $\Psi = I$ (the identity mapping $I(R) = R$). On the other hand, $\sigma(\Phi(T_n)) = \sigma(\phi(T_n)) = \sigma(I(T_n))$ follows from (1) and the spectrality of ϕ. The conclusion $\sigma(\Phi(X)) = \sigma(I(X)) = \sigma(X)$ is now a consequence of (2).

2.8 Definitions. *The set l_0 consists of all sequences a such that $a_n = 0$ except for finitely many non-negative values of n. We say that $P \varepsilon \mathfrak{P}$ if $P = \sum a_n I^n$ for some a in l_0 (recall that $I^n(\theta) = \theta^n$); clearly $a_n = P^{(n)}(0)/n!$ and $\mathfrak{P}$ is the family of all polynomials. In case J belongs to some Banach algebra $\mathfrak{E}$, we denote by $[J;\mathfrak{E}]$ the mapping $P \to \sum (P^{(n)}(0)/n!) J^n$ of $\mathfrak{P}$ into $\mathfrak{E}$; the closure in $\mathfrak{E}$ of the range of $[J;\mathfrak{E}]$ is the subalgebra $\mathfrak{E}[J]$ generated by J.*

2.9 Lemma. *Suppose J is a member of a Banach algebra $\mathfrak{E}$ such that $\sigma(J)$ is an infinite set $\mathfrak{S}$. If $\psi = [J;\mathfrak{E}]$, then ψ is an isomorphism of $\mathfrak{P}$ into $\mathfrak{E}$, and $\psi(I) = J$. Any homomorphism ψ_1 of $\mathfrak{P}$ into $\mathfrak{E}$ which maps I on J is identical to ψ.*

Proof. If $P \varepsilon \mathfrak{P}$, then $P = \sum a_n I^n$ and $\psi(P) = \sum a_n J^n$ for some a in l_0. By the Dunford mapping theorem ([3], 2.8 and 2.9)

(3) $$\sigma(\psi(P)) = P(\sigma(J)) = P(\mathfrak{S}) \qquad (\text{when } P \varepsilon \mathfrak{P}).$$

It is easily checked that ψ is a homomorphism. Suppose $\psi(P) = \mathsf{O}$; this implies (taking $\{0\} = \sigma(\mathsf{O})$ and (3) into account) $\{0\} = \sigma(\psi(P)) = P(\mathfrak{S})$. Hence the polynomial P vanishes at all points of the infinite (and bounded) set $\mathfrak{S}$; consequently $P = \mathsf{O}$. This proves that the linear transformation ψ

is an isomorphism. We conclude by observing that the homomorphism ψ_1 satisfies $\psi_1(\sum a_n I^n) = \sum a_n[\psi_1(I)]^n$, so that $\psi_1 = \psi$ when $\psi_1(I) = J$.

2.10 LEMMA. *Let $\mathfrak{E}$, J, and $\mathfrak{S}$ be as in 2.9. There exists a spectral homomorphism Φ of $\mathfrak{E}[J]$ into $\boldsymbol{C}(\mathfrak{S})$ which maps J on I. If $\psi = [J;\mathfrak{E}]$, then $P = \Phi(\psi(P))$ when $P \varepsilon \mathfrak{P}$.*

Proof. If $P \varepsilon \mathfrak{P}$ and $T = \psi(P)$, then $\sigma(T) = P(\mathfrak{S}) = \sigma(P)$ (by (3), $P \varepsilon \boldsymbol{C}(\mathfrak{S})$, and 2.1). Hence the inverse ϕ of ψ is a spectral homomorphism of the ring $\psi(\mathfrak{P}) = \mathfrak{T}$ into $\boldsymbol{C}(\mathfrak{S})$. Since $\mathfrak{T}$ is dense in the Banach algebra $\mathfrak{E}[J]$, the conclusion now follows from 2.7.

2.11 *Remark.* From 2.3 follows that the mapping Φ of 2.10 is a continuous homomorphism of $\mathfrak{E}[J]$ into $\boldsymbol{C}(\mathfrak{S})$ mapping J on I. We will now show that it is the only such mapping. Take $\iota = 1, 2$ and let Φ_ι be continuous homomorphisms of $\mathfrak{E}[J]$ into $\boldsymbol{C}(\mathfrak{S})$ such that $\Phi_\iota(J) = I$: note that Φ_1 and Φ_2 coincide on the set $\psi(\mathfrak{P}) = \{\sum a_n J^n : a \varepsilon l_0\}$ (since $\Phi_\iota(\sum a_n J^n) = \sum a_n[\Phi_\iota(J)]^n = \sum a_n I^n$); but Φ_1 and Φ_2, being also continuous on the closure $\mathfrak{E}[J]$ of $\psi(\mathfrak{P})$, must consequently coincide on $\mathfrak{E}[J]$ (cf. [7]; 7 F). The existence of such a mapping could equally well have been obtained by appealing to Gelfand's theory of maximal ideals ([7], 23 E).

2.12 DEFINITION. *Suppose J is a member of a Banach algebra $\mathfrak{E}$ such that $\sigma(J)$ is an infinite set. The "Gelfand-transformation" $G(\mathfrak{E}, J)$ is the (unique) continuous homomorphism of $\mathfrak{E}[J]$ into $\boldsymbol{C}(\sigma(J))$ which maps J on I.*

2.13 *Remark.* If Φ is the Gelfand-transformation $G(\mathfrak{E}, J)$, then (by 2.11 and 2.10) Φ is a spectral homomorphism of $\mathfrak{E}[J]$ into $\boldsymbol{C}(\sigma(J))$ such that, if $\psi = [J;\mathfrak{E}]$, then

$$\Phi(\psi(P)) = P \qquad \text{for all } P \text{ in } \mathfrak{P}.$$

In other words, $G(\mathfrak{E}, J)$ is the continuous extension to $\mathfrak{E}[J]$ of the inverse of $[J;\mathfrak{E}]$.

3. The Banach algebras (*AC*) and (*BV*). The set (BV) of functions of bounded variation on $[-\pi, \pi]$ becomes a Banach algebra under the norm

$$\|B\|_b = \|B\|_\infty + \operatorname{var} B \qquad (B \varepsilon (BV)) \tag{4}$$

(cf. [8], p. 448), where $\operatorname{var} B$ designates the total variation of B on $[-\pi, \pi]$ and

$$\|B\|_\infty = \sup\{|B(\theta)| : |\theta| \leqq \pi\}. \tag{5}$$

Banach and Mazur ([2], p. 101) make (BV) into a Banach algebra by adopting the norm $\| B \|_v$ obtained from (4) by replacing $\| B \|_\infty$ by $B(-\pi)$; note that $\| B \|_v \leqq \| B \|_b \leqq 2 \| B \|_v$, so that the resulting topologies are equivalent. The set (AC) of absolutely continuous members of (BV) forms a Banach algebra under the norm of (BV) (cf. [1], p. 194). It will be implied henceforth that (4) defines the norm for both (BV) and (AC). That (AC) forms then a separable space, follows from 3.1.

3.1 LEMMA. *If $A \varepsilon \mathfrak{F} = (AC)$, then there exists a sequence $\{P_n\}$ of members of $\mathfrak{P}$ such that $A = \mathfrak{F} \operatorname{lm} P_n$.*

Proof. Let P_n be the n-th Bernstein polynomial of A. Since A is continous on $[-\pi, \pi]$, we see from ([9], p. 5) that $\{P_n\}$ converges uniformly to A; therefore $\lim \| A - P_n \|_\infty = 0$. But $A \varepsilon (AC)$ and consequently (cf. [10], Satz 7) $\lim \operatorname{var} (A - P_n) = 0$. This concludes the proof, since (4) defines the norm of $\mathfrak{F}$.

4. The main results. Choose for p a fixed value $p > 1$, and denote by Ψ the mapping $F \rightarrow \langle \Lambda F \rangle_p$ defined in the introduction. In a recent paper [13], Stečkin has shown that Ψ maps (BV) into the Banach algebra $\mathfrak{E}$ of bounded operators on l_p. From now on, $\mathfrak{R}$ will denote either one af the two Banach algebras (BV) and (AC); thus, $A = \mathfrak{R} \operatorname{lm} P_n$ means that $\lim \| A - P_n \|_b = 0$. Henceforth, $\mathfrak{C} = \boldsymbol{C}([-\pi, \pi])$; thus, $A = \mathfrak{C} \operatorname{lm} P_n$ means that $\lim \| A - P_n \|_\infty = 0$ (see § 2 and (5)).

4.1 LEMMA. *If $F_n \varepsilon \mathfrak{R}$ and $A = \mathfrak{R} \operatorname{lm} F_n$, then $A = \mathfrak{C} \operatorname{lm} F_n$. Suppose moreover that $T = \mathfrak{E} \operatorname{lm} \Psi(F_n)$; then $T = \Psi(A)$.*

Proof. Set $f_n = A - F_n$ and note that $\| f_n \|_\infty \leqq \| f_n \|_b$; the conclusion $A = \mathfrak{C} \operatorname{lm} F_n$ follows. The completeness of the space $\mathfrak{R}$ necessitates that $A \varepsilon \mathfrak{R} \subset (BV)$; the conclusion $T = \Psi(A)$ is now given by 4.3 in [6].

4.2 THEOREM. *The mapping Ψ is a continuous isomorphism of $\mathfrak{R}$ into $\mathfrak{E}$ such that $\Psi(I) = H$, where H is characterized by the Laurent matrix $(a_{n-\nu})$ with $a_m = i(-1)^m/m$ and $a_0 = 0$.*

Proof. It was shown in [6] that $\Psi(I) = H$ (the notation used there is $I_\#$ instead of H), and that Ψ is an isomorphism of the Banach space $\mathfrak{R}$ into the Banach space $\mathfrak{E}$. It will therefore suffice to show that Ψ is a closed operator (see [5], p. 30). To that effect, suppose $F_n \varepsilon \mathfrak{R}$, $A = \mathfrak{R} \operatorname{lm} F_n$, and $T = \mathfrak{E} \operatorname{lm} \Psi(F_n)$; the conclusion $T = \Psi(A)$ is given by 4.1.

4.3 THEOREM. *The isomorphism Ψ is the only continuous homomor-*

phism of $\mathfrak{J} = (AC)$ *into* $\mathfrak{C}$ *such that* I *is mapped on* H. *The image* $\Psi(\mathfrak{J})$ *is dense in* $\mathfrak{C}[H]$, *and the Gelfand-transformation* $G(\mathfrak{C}, H)$ *coincides on* $\Psi(\mathfrak{J})$ *with the inverse mapping of* Ψ. *Moreover, the properties* (i)-(iii) *of the introduction are satisfied.*

Proof. Suppose $\iota = 1, 2$ and let Ψ_ι be continuous homomorphisms of $\mathfrak{J}$ into $\mathfrak{C}$ such that $\Psi_\iota(I) = H$. The restrictions ψ_ι of Ψ_ι to $\mathfrak{P}$ are homomorphisms of $\mathfrak{P}$ into $\mathfrak{C}$ such that $\psi_\iota(I) = H$; having proved in ([6], 6.4) that $\sigma(H) = [-\pi, \pi]$, we can infer from 2.9 that

$$\psi_\iota = [H; \mathfrak{C}] = \psi \quad \text{and} \quad \Psi_\iota(P) = \psi(P) \qquad (\text{when } P \in \mathfrak{P}). \tag{6}$$

If $A \in \mathfrak{J}$, then (by 3.1) there exists a sequence $\{P_n\}$ satisfying

$$A = \mathfrak{J} \operatorname{lm} P_n \qquad (n \to \infty, P_n \in \mathfrak{P}). \tag{7}$$

From the continuity of Ψ_ι follows that $\Psi_\iota(A) = \mathfrak{C} \operatorname{lm} \Psi_\iota(P_n)$. This enables us to derive from (6) that

$$\Psi_\iota(A) = \mathfrak{C} \operatorname{lm} \psi(P_n) \qquad (n \to \infty). \tag{8}$$

Thus $\Psi_1(A) = \Psi_2(A)$, and $\Psi_1 = \Psi_2$. Hence, there is at most one continuous homomorphism of $\mathfrak{J}$ into $\mathfrak{C}$ which maps I on H; from 4.2 now follows that Ψ is the only such homomorphism. In the following, (6) and (8) should be viewed in the light of $\Psi_\iota = \Psi$.

Next, observe that $\psi(P_n)$ is in the closure $\mathfrak{C}[H]$ of $\psi(\mathfrak{P})$ (by 2.8), so that $\Psi(A) \in \mathfrak{C}[H]$ (from (8)). This implies $\Psi(\mathfrak{J}) \subset \mathfrak{C}[H]$. Since $\psi(\mathfrak{P}) \subset \Psi(\mathfrak{J})$ (by (6) and $\mathfrak{P} \subset \mathfrak{J}$), the denseness of $\psi(\mathfrak{P})$ in $\mathfrak{C}[H]$ now necessitates that

$$\Psi(\mathfrak{J}) \text{ is dense in } \mathfrak{C}[H].$$

We now turn to the conclusions involving $G(\mathfrak{C}, H)$. Again referring to [6] for the result $\sigma(H) = [-\pi, \pi]$, we see from 2.12 that the Gelfand-transformation $G(\mathfrak{C}, H)$ is a continuous homomorphism Φ of $\mathfrak{C}[H]$ into $\mathfrak{K} = \boldsymbol{C}([-\pi, \pi])$. A successive application of (8) with the continuity of Φ, and 2.13 (with $\psi = [H; \mathfrak{C}]$) yields

$$\Phi(\Psi(A)) = \mathfrak{K} \operatorname{lm} \Phi(\psi(P_n)) = \mathfrak{K} \operatorname{lm} P_n \qquad (n \to \infty).$$

This result, combined with the consequence $A = \mathfrak{K} \operatorname{lm} P_n$ of (7) (see 4.1), shows that $\Phi(\Psi(A)) = A$. Accordingly, if $T = \Psi(A)$ and if ϕ is the inverse of the mapping Ψ, then $\Phi(T) = A = \phi(T)$, which states that Φ coincides with ϕ on $\Psi(\mathfrak{J})$. On the other hand, $\Phi(T) = A \in \mathfrak{J}$ and 2.4 shows that (iv)-(v) are satisfied, in consequence of Φ being a spectral homomorphism

of $\mathfrak{X} = \mathfrak{C}[H]$ into $\boldsymbol{C}(\mathfrak{S}), \mathfrak{S} = [-\pi, \pi]$ (see 2.13). Therefore (i)-(iii) hold, and the proof is completed.

4.4 *Remark.* Suppose $A \varepsilon (AC)$ and set $T = \Psi(A)$. In the preceding paragraph, we have pointed out that the relations 2.4 (iv) are satisfied; hence $\| A \|_\infty \leqq \| T \|$. In case $p = 2$, then $\| A \|_\infty = \| T \|$. This was proved by F. Riesz [12] and can in the present context be derived as follows. The operator T is a member of the abelian Hilbert space $\mathfrak{C}[H]$ and the same holds for its adjoint T^*; therefore T is a normal operator. But then $\| T \| = \sup \{ | \lambda | : \lambda \varepsilon \sigma(T) \} = \sup \{ | \lambda | : \lambda \varepsilon A(\mathfrak{S}) \} = \| A \|_\infty$; the first equality holds for any normal operator, the second follows from $\sigma(T) = A(\mathfrak{S})$, the third equality is obvious.

4.5 *Remark.* We here supply a few details connected with 2.4 and 2.6. It should be kept in mind that l_1 is a Banach algebra $\mathfrak{C}$ having a member J such that $\sigma(J) = \Gamma_1 = \{\lambda : |\lambda| = 1\}$, and l_1 is the subalgebra $\mathfrak{C}[J]$ generated in $\mathfrak{C}$ by J. The inverse ∇ of the mapping $f \rightarrow \Delta f$ is an isomorphism of l_1 onto $\mathfrak{U}$ (the symbols Δf and $\mathfrak{U}$ are defined in 1.1). If we set $\mathfrak{X} = l_1$, $\mathfrak{F} = \mathfrak{U}$ and $\mathfrak{S} = \Gamma_1$, then (iv)-(v) hold when $T \varepsilon \mathfrak{X}$ and $A = \nabla(T)$. This can be seen as follows. Let $\Phi(T)$ be the function defined on Γ_1 by $f(\lambda) = \sum T_n \lambda^n$; clearly Φ is a continuous homomorphism of $l_1 = \mathfrak{C}[J]$ into $\boldsymbol{C}(\mathfrak{S})$, and 2.12 now shows that Φ is the Gelfand-transformation $G(\mathfrak{C}, J)$. Accordingly, Φ is spectral; the conclusion is now obtained from 2.4 and the fact that $\nabla(T)$ coincides with $\Phi(T)$ on Γ_1.

Consequently, $f^{-1} \varepsilon \mathfrak{U}$ when $f \varepsilon \mathfrak{U}$ and provided $0 \notin f(\Gamma_1)$. This conclusion can be used in the otherwise obvious derivation of (i′)-(iii′) from (i)-(iii). Moreover, we have just seen that $\sigma(T) = (\Phi(T))(\Gamma_1)$; since the isomorphism V of 2.6 satisfies $(V(T))(\theta) = (\Phi(T))(e^{i\theta})$, we can conclude that V is a spectral isomorphism of $\mathfrak{X} = l_1$ onto the ring of all absolutely convergent Fourier series. In view of these facts, the classical Wiener theorems ([7], pp. 72-73) now appear as consequences of 2.4.

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A NOTE ON THE CLASS-NUMBERS OF ALGEBRAIC NUMBER FIELDS.*

By N. C. Ankeny, R. Brauer and S. Chowla.

Let F be an algebraic number field of finite degree n over the field P of rational numbers. Denote by $h(F)$ and $d(F)$ the class-number and discriminant of F respectively. According to Minkowski each class of ideals of F contains an ideal of normal at most $|d(F)|^{\frac{1}{2}}$. Landau [1] used this result to deduce

$$h(F) < c_1 \,|\, d(F) \,|^{\frac{1}{2}} (\log |\, d(F) \,|)^{n-1}$$

where c_1 is a constant depending on n alone. If ϵ is a given positive number, this implies

$$h(F) < c_2 \,|\, d(F) \,|^{\frac{1}{2}+\epsilon} \tag{1}$$

where the constant c_2 depends on n and ϵ.

We shall show that, for suitable fields F, the rather rough estimate (1) is actually remarkably sharp. Indeed, given any positive integer $n \geqq 2$, let r_1 and r_2 be any two non-negative integers such that $r_1 + 2r_2 = n$. We shall prove that for every $\epsilon > 0$ there exist infinitely many fields F which have exactly r_1 real and $2r_2$ imaginary conjugate fields and are such that

$$h(F) > |\, d(F) \,|^{\frac{1}{2}-\epsilon} \tag{2}$$

holds.

For the proof, we use the following result of R. Brauer [2] which confirmed a conjecture of C. L. Siegel. For all fields F of given degree n with sufficiently large $|\, d(F) \,|$, we have

$$h(F)R(F) > |\, d(F) \,|^{\frac{1}{2}-\delta} \tag{3}$$

where $R(F)$ is the regulator of F and $\delta > 0$ a given arbitrary constant. For $n = 2$, this had already been proved by Siegel.

We can immediately settle the case $n = 2$ as follows. If F is an imaginary quadratic extension of the field P of rationals, then (2) holds

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for all fields F with $|d(F)|$ exceeding a certain limit depending on ϵ alone. To construct real quadratic fields with the property in question, let F be generated by $[m^2+1]^{\frac{1}{2}}$, where we select the rational integer $m \geqq 2$ so that m^2+1 is square free. We recall the well-known result of Estermann [4] that m^2+1 is square-free for infinitely many choices of m. Since $E = m + [m^2+1]^{\frac{1}{2}}$ is a unit contained in F and since $d(F) = m^2+1$ or $d(F) = 4(m^2+1)$, we have

$$R(F) \leqq \log(m + [m^2+1]^{\frac{1}{2}}) \leqq \log(2[m^2+1]^{\frac{1}{2}}) \leqq \log(m^2+1) \leqq \log d(F).$$

So (2) follows from (3) for the fields $F = P([m^2+1]^{\frac{1}{2}})$, where m^2+1 is square-free and sufficiently large.

Section 1. n is any fixed integer $\geqq 3$, N is an arbitrary positive integer that is taken to be sufficiently large.

Let $a_1, a_2, a_3, \cdots, a_{n-1}$ be arbitrary fixed distinct integers, and take $a_n = N$. We define

$$f_N(x) = \prod_{j=1}^{n} (x - a_j) + 1.$$

We prove (see the references [5], [6]).

LEMMA 1. *If N is sufficiently large, $f_N(x)$ enjoys the following properties:*

(4) $f_N(a_j) = 1 \qquad (1 \leqq j \leqq n).$

(5) $f_N(x) = 0$ *has n distinct real roots:* $\theta_N^{(1)}, \theta_N^{(2)}, \cdots, \theta_N^{(n)}$.

For a suitable arrangement of these roots there exist constants $b_i \neq 0$ such that, for $N \to \infty$,

(6) $N(\theta_N^{(i)} - a_i) \to b_i \neq 0 \quad (1 \leqq i \leqq n-1), \qquad \theta_N^{(n)} = N + O(N^{-1}).$

(7) *$f_N(x)$ is an irreducible equation of degree n in the field of rational numbers.*

Proof. To prove that the roots are real, we note that $f(a_j + \frac{1}{2})$ and $f(a_j - \frac{1}{2})$, $j = 1, \cdots, n$ are of different sign for N sufficiently large.

(4) is clear from the definition of $f_N(x)$. Next, *as* $N \to +\infty$,

$$-(1/N) f_N(x) \to \prod_{i=1}^{n-1} (x - a_i).$$

Because of the continuity of the roots as functions of the parameter N, each of the a_i, $(i \neq n)$, is the limit of a root $\theta_N^{(i)}$ of $f_N(x)$. Also

$$(\theta_N^{(i)} - a_i)(\theta_N^{(i)} - N)Z + 1 = 0$$

where, as $N \to \infty$,

$$Z = \prod_{\mu \neq i,\, \mu \neq n} (\theta_N^{(i)} - a_\mu) \to \prod_{\mu \neq i,\, n} (a_i - a_\mu) \neq 0.$$

Hence $N(\theta_N^{(i)} - a_i) \to \prod_{\mu \neq i,\, n} (a_i - a_\mu)^{-1}$, and this limit is a constant $b_i \neq 0$. In particular, $\theta_N^{(i)} = a_i + O(N^{-1})$ $(1 \leqq i \leqq n-1)$. Since $\sum_{i=1}^{n} \theta_N^{(i)} = \sum_{i=1}^{n} a_i$, it follows that $\theta_N^{(n)} = a_n + O(N^{-1}) = N + O(N^{-1})$. The above equations prove (6).

Next let us assume that there are infinitely many N for which $f_N(x)$ is reducible and set $f_N(x) = g_{1,N}(x) g_{2,N}(x)$ where $g_{1,N}(x)$ and $g_{2,N}(x)$ have integral rational coefficients. By (5), we may always select $g_{1,N}(x)$ such that $\theta_N^{(n)}$ is not a root of $g_{1,N}(x)$. If $\theta_N^{(i)}$ is a root of $g_{1,N}(x)$, all the conjugates of $N - \theta_N^{(i)} = a_n - \theta_N^{(i)}$ will be larger than 1 for sufficiently large N. This is a contradiction, since (4) shows that $a_n - \theta_N^{(i)}$ is a unit and hence has norm ± 1. This completes the proof of the lemma.

We note that the coefficients of $f_N(x)$ are linear functions of N. Hence $\operatorname{discr}(f_N(x)) = g(N)$, where $g(N)$ is a polynomial in N whose coefficients are rational integers. On considering the order of magnitude of

$$g(N) = \prod_{i > j} (\theta_N^{(i)} - \theta_N^{(j)})^2,$$

we see that $g(N)$ is of degree $2(n-1)$.

The following lemma is proved for general polynomials $g(x)$ and later specialized to the $g(x)$ above. We prove

LEMMA 2. *Let $g(x)$ be a polynomial with integral coefficients of degree $s > 0$. Let m be the greatest common divisor of the values of $g(x)$ for integral x. Let $\rho \geqq 3$ be a fixed number, and let U be a number chosen sufficiently large. If U^* denotes the number of integers N for which $U < N \leqq 2U$ and $g(N)/m$ has all prime factors greater than $V = (\frac{1}{4} \log U)^\rho$, then*

(8) $$U^* > cU/(\log\log U)^s.$$

Here c is an absolute positive constant depending only on the coefficients of $g(x)$ and on ρ.

The proof of Lemma 2 is in turn based on the following Lemma 3 which is a recent theorem of de Bruijn [3]. For a simple proof, see the paper of W. E. Briggs and S. Chowla [7].

LEMMA 3. *Let $f(x,y)$ denote the number of positive integers $\leqq x$ all of whose prime factors are $\leqq y$. Then*

$$f(x, (\log x)^{\rho}) = O(x^{1-1/(2\rho)}) \tag{9}$$

where $\rho > 2$ and the constant implied in the O-symbol depends on ρ alone.

Section 2. In this section, we shall prove Lemma 2.

For a positive integer d, let $\lambda(d)$ denote the number of solutions of $g(x) \equiv 0 (\bmod d)$ with $0 \leqq x < d$. In the following, the letter p will always denote a prime number. We have

$$\lambda(p) \leqq s \text{ if } p \nmid m, \qquad \lambda(d) \leqq d \text{ for all } d. \tag{10}$$

Denote by $\nu(d)$ the number of distinct prime factors of d. Then, for any fixed $\epsilon > 0$,

$$s^{\nu(d)} = O(d^{\epsilon}). \tag{11}$$

To prove this, take $s > 1$ and note that

$$s^{\nu(d)} = 2^{\nu(d) \log s/\log 2} \leqq \{\tau(d)\}^{\log s/\log 2}$$

where $\tau(d)$ is the number of divisors of d. Since (see, for example, Landau's Vorlesungen ü. Zahlentheorie)

$$\tau(d) = O(d^{\theta}), \qquad \theta = \epsilon \log 2/\log s,$$

(11) now follows. We shall use the symbol $\mu(t)$ for the well-known Möbius function so that

$$\mu(t) = 0 \text{ if } p^2 \mid t \text{ for some prime } p, \qquad \mu(t) = (-1)^{\nu(t)} \text{ for square-free } t.$$

As is well-known,

$$\lambda(ab) = \lambda(a)\lambda(b) \text{ for } (a,b) = 1. \tag{12}$$

We also need

$$\lambda(d) = O(d^{\epsilon}) \qquad (d \to \infty), \tag{13}$$

if d is a multiple of m such that d/m is square-free. This can be seen easily (actually (13) holds for all d).

We now introduce the function $Q(d)$ defined as follows:

(i) $Q(d) = \mu(d)$ if $1 \leqq d \leqq U^{\frac{1}{4}}$.

(ii) $Q(d) = \mu(d)$ if $U^{\frac{1}{4}} < d \leqq U^{\frac{1}{2}}$ and $\nu(d)$ odd.

(iii) $Q(d) = 0$ if $U^{\frac{1}{4}} < d \leqq U^{\frac{1}{2}}$ and $\nu(d)$ even.

(iv) $Q(d) = 0$ if $d > U^{\frac{1}{2}}$.

Observe that $Q(d) = 0$, if $\mu(d) = 0$ and that $Q(d) = 1$, if and only if $d \leqq U^{\frac{1}{4}}$, d square-free, $\nu(d)$ even. Further, set $Q(d) = 0$, if d is not integral.

LEMMA 4. *If $k > 0$ is an integer and if $S(k) = \sum_{d \mid k} Q(d/m)$, then*

(α) *$S(k) = 1$ if $k/m = k_0$ is integral and not divisible by a prime $\leqq U^{\frac{1}{2}}$;*

(β) *$S(k) \leqq 0$ in all other cases.*

Proof. If $k \not\equiv 0 \pmod{m}$, obviously $S(k) = 0$. Suppose that $k_0 = k/m$ is integral. Then

$$S(k) = \sum_{t \mid k_0} Q(t). \tag{14}$$

If k is not divisible by primes $\leqq U^{\frac{1}{2}}$, then $S(k) = Q(1) = 1$. Suppose that k_0 is divisible by a prime $p \leqq U^{\frac{1}{2}}$ and choose p as the least such prime. Since it suffices to take square-free t in (14), we can arrange these t in pairs h and hp where h is a divisor of k_0 which is prime to p. In order to prove (β) it will be sufficient to show that $Q(h) + Q(hp) \leqq 0$. If this was not so, we must have either

$$Q(hp) = 1, \quad Q(h) \geqq 0 \quad \text{or} \quad Q(h) = 1, \quad Q(hp) = 0.$$

Since $Q(x) = 1$ implies that $x \leqq U^{\frac{1}{4}}$ and that x is square-free, it follows easily from $Q(hp) = 1$ that $Q(h) = -1$ and hence the former case is impossible. Suppose then that $Q(h) = 1$, $Q(hp) = 0$. Then $h \leqq U^{\frac{1}{4}}$, h square-free and $\nu(h)$ even. In order to have $Q(hp) = 0$, we must have $hp > U^{\frac{1}{2}}$ and hence $p > U^{\frac{1}{4}}$. Then every divisor $h \neq 1$ of k_0 with $\nu(h)$ even would exceed $U^{\frac{1}{2}}$ and this would imply $Q(h) = 0$. Hence $h = 1$. In this case $Q(hp) = Q(p) = -1$. This is a contradiction and the lemma is proved.

Suppose that U is so large that $V = (\frac{1}{4} \log U)^{\rho} \leqq U^{\frac{1}{2}}$ and that $V \geqq 2m$. Let U^* denote the number of N with $U < N \leqq 2U$ for which $k_1 = g(N)/m$ is not divisible by primes $\leqq V$. Then if $(g(N), [V]!)$ denotes the g.c.d. of $g(N)$ and $[V]!$.

$$U^* \geqq \sum_{U < N \leqq 2U} S((g(N), [V]!)). \tag{15}$$

Indeed, by Lemma 4 all terms here are either $\leqq 0$ or have the value 1, and the latter case arises only if $k = (g(N), [V]!) = mk_0$ where k_0 has no prime factors $\leqq U^{1/2}$. In this case, $g(N)/m$ cannot have a prime factor $\leqq V$ and all such N are certainly counted by U^*.

Let g_d denote the largest prime dividing any integer $d > 1$ and set $g_1 = 1$. Then

$$U^* \geqq \sum_{U<N\leqq 2U} \sum_{d \mid (g(N),[V]!)} Q(d/m) = \sum_d Q(d/m) \sum_N 1.$$

Here, d ranges over all positive integers such that

(a) $g_d \leqq V$; (b) $d \equiv 0 \pmod{m}$; (c) d/m is square-free;

while N ranges over all integers with $U < N \leqq 2U$ and $g(N) \equiv 0 \pmod{d}$. Thus,

$$U^* \geqq U \sum_d Q(d/m)\lambda(d)/d - \sum_d |Q(d/m)| \lambda(d).$$

Let $\epsilon < 1/(2\rho)$ be a positive constant. Using (13) and recalling that $Q(d) = 0$ for $d > U^{1/2}$, we obtain

$$U^* \geqq U \sum_d Q(d/m)\lambda(d)/d - O(U^{1/2+\epsilon}). \tag{16}$$

If we replace $Q(d/m)$ by $\mu(d/m)$, the error term is

$$\Big|\sum_d Q(d/m)\lambda(d)/d - \mu(d/m)\lambda(d)/d\Big| \leqq \sum_t \lambda(t)/t \leqq O(\sum_t t^{-1+\epsilon}) \tag{17}$$

where t ranges over all integers with $t > mU^{1/4}$ for which $g_t \leqq V$. The convergence of the sum on the right will become evident from the following argument.

For $x > mU^{1/4} \geqq U^{1/4}$, we have $V = (\frac{1}{4}\log U)^\rho < (\log x)^\rho$. If $f(x,y)$ is the expression introduced in Lemma 3, this implies $f(x,V) \leqq f(x, (\log x)^\rho) = O(x^{1-1/(2\rho)})$. Now, if n ranges over the integers larger than $mU^{1/4}$, we find by partial summation

$$\sum_t t^{-1+\epsilon} = \sum_n n^{-1+\epsilon}(f(n,V) - f(n-1,V))$$

$$\leqq (1-\epsilon) \int_{mU^{1/4}}^{\infty} f(x,V)x^{-2+\epsilon}dx = O\Big(\int_{mU^{1/4}}^{\infty} x^{-1-1/(2\rho)+\epsilon}dx\Big),$$

$$\sum_t t^{-1+\epsilon} = O(U^{-1/(8\rho)+\epsilon/4}). \tag{18}$$

Since $\epsilon < 1/(2\rho)$, the exponent of U is negative. Hence, by substituting (17) and (18) in (16), we have

$$U^* \geqq U \sum_d \mu(d/m)\lambda(d)/d - O(U^{1-\gamma}) \tag{19}$$

where γ is a positive constant and where d still ranges over the integers which satisfy the conditions (a), (b), and (c).

Set $d = m\alpha\beta$ with $(\alpha, m) = 1, \beta \mid m$. Then

$$\mu(d/m)\lambda(d)/d = \mu(\alpha)\lambda(\alpha)/\alpha\, \mu(\beta)\lambda(\beta m)/\beta m)$$

and hence

$$(20) \qquad \sum_d \mu(d/m)\lambda(d)/d = \sum_\alpha \mu(\alpha)\,(\lambda(\alpha)/\alpha) \sum_{\beta \mid m} \mu(\beta)\lambda(\beta m)/(\beta m).$$

In the first sum on the right, α ranges over the integers prime to m for which $g_\alpha \leqq V$. Therefore

$$\sum_\alpha \mu(\alpha)\lambda(\alpha)/\alpha = \prod_{p \leqq V,\, p \nmid m} (1 - \lambda(p)/p).$$

If β is a fixed divisor of m, let $m_1(\beta)$ denote the product of those prime powers of m which are prime to β and set $m = m_1(\beta) m_2(\beta)$. Then $\lambda(\beta m) = \lambda(m_1(\beta))\lambda(\beta m_2(\beta))$. Since $m_1(\beta) \mid m$, we have $\lambda(m_1(\beta)) = m_1(\beta)$. Thus,

$$\sum_\beta \mu(\beta)\lambda(\beta m)/(\beta m) = \sum_\beta \mu(\beta)\lambda(\beta m_2(\beta))/(\beta m_2(\beta)).$$

If $\beta = \beta'\beta''$ with $(\beta', \beta'') = 1$, then $m_2(\beta) = m_2(\beta')m_2(\beta'')$ and

$$\lambda(\beta m_2(\beta)) = \lambda(\beta' m_2(\beta'))\lambda(\beta'' m_2(\beta'')).$$

Hence

$$\sum_\beta \mu(\beta)\lambda(\beta m)/(\beta m) = \prod_{p \mid m} (1 - \lambda(p m_2(p))/(p m_2(p))).$$

Now, $pm_2(p)$ does not divide m and, consequently, $\lambda(pm_2(p)) \neq pm_2(p)$. It follows that the last product is a positive constant c_3. This shows that

$$\begin{aligned}\sum_d \mu(d/m)\lambda(d)/d &= c_3 \prod_{p \leqq V,\, p \nmid m} (1 - \lambda(p)/p) \\ &\geqq c_3 \prod_{p \leqq s} (1 - (p-1)/p) \prod_{s < p \leqq V} (1 - s/p).\end{aligned}$$

By Mertens' theorem on prime numbers, we have

$$(21) \qquad \sum_d \mu(d/m)\lambda(d)/d \geqq c_4 (\log V)^{-s} \geqq c_5 (\log\log U)^{-s}$$

where c_4 and c_5 are also positive constants, independent of U. On combining (19) and (21), we obtain $U^* \geqq cU(\log\log U)^{-s}$ with a positive constant c, and this proves Lemma 2.

Section 3. We shall now prove that there are infinitely many totally real algebraic number fields of degree $n \geqq 3$ over the rationals, and such that (2) holds.

Writing $F_N = P(\theta_N^{(1)})$ we prove

LEMMA 5. *There exist an infinite set of N for which* $|d(F_N)| > (\frac{1}{4}\log\frac{1}{2}N)^\rho$ *where ρ is an arbitrary but fixed positive integer.*

We refer back to Lemma 2. Let $S(U)$ denote the set of N which satisfy the hypotheses of Lemma 2. Now we observe that $d(F_N)$ divides the discriminant $g(N)$ of $f_N(x)$. For $N \varepsilon S(U)$ it follows that if $|d(F_N)| \leqq (\frac{1}{4}\log\frac{1}{2}N)^\rho$, then $|d(F_N)| \leqq (\frac{1}{4}\log U)^p$ and $d(F_N) \mid m$. Thus if Lemma 5 were not true, we would have $d(F_N) \mid m$ for all $N \varepsilon S(U)$, U sufficiently large. It is a well-known theorem of Minkowski that there are only a finite number of algebraic number fields whose discriminant has a presecribed value. Hence there are only a finite number of fields whose discriminant divides m.

By Lemma 2 as U increases there is an increasing number of $N \varepsilon S(U)$. Let us select U sufficiently large so that at least one of the fields, say F', contains $\theta_N^{(1)}$ for at least $(n-1)!+1$ values of N belonging to $S(U)$. Consider the n isomorphisms of F' onto its conjugate fields. Using Dirichlet's chest of drawers principle we shall be able to select N_1 and N_2 with $U < N_1 < N_2 \leqq 2U$ such that $\theta_{N_1}^{(1)}$, $\theta_{N_2}^{(1)} \varepsilon F'$ and that each of the n isomorphisms of F' maps $\theta_{N_1}^{(1)}$, $\theta_{N_2}^{(1)} \varepsilon F'$ on $\theta_{N_1}^{(j)}$, $\theta_{N_2}^{(j)}$ respectively, with the same j. Write $\phi^{(j)} = \theta_{N_1}^{(j)} - \theta_{N_2}^{(j)}$. Now $\phi^{(1)} \varepsilon F'$ and the $\phi^{(j)}$ are the n conjugates of $\phi^{(1)}$. For $N_1 \neq N_2$ we have $\theta_{N_1}^{(1)} \neq \theta_{N_2}^{(1)}$ since we have the roots of two different irreducible equations. Hence $\phi^{(1)} \neq 0$. Now by Lemma 1, $|\phi^{(j)}| < c_6/N_1 \leqq c_6/U$ for $j = 1, 2, 3, \cdots, n-1$.[1] Further $|\phi^{(n)}| < N_2 + c_7 \leqq 2U + c_7$. Hence

$$|N_{F',P}(\phi^{(1)})| = |\prod_{j=1}^{n} \phi^{(j)}| < c_8 U^{-n+2}.$$

If U is sufficiently large, this implies $N_{F',P}(\phi^{(1)}) = 0$, since the norm of $\phi^{(1)}$ is an integer. Hence $\phi^{(1)} = 0$, a contradiction. This concludes the proof of Lemma 5.

We now prove

THEOREM 1. *For given $\epsilon > 0$, there exists an infinity of totally real fields F of given degree n such that the class-number $h(F)$ satisfies the inequality* $h(F) \geqq |d(F)|^{\frac{1}{2}-\epsilon}$, *where $d(F)$ is the discriminant of F.*

Proof. We show that we may take F as one of the infinitely many fields F_N where N satisfies Lemma 5 for a suitable choice of ρ and is sufficiently large. It is clear from the definition of $f_N(x)$ that

$$\theta_N^{(1)} - a_1, \theta_N^{(1)} - a_2, \cdots, \theta_N^{(1)} - a_{n-1}$$

[1] We denote by $c_6, c_7 \cdots$ positive constants which may depend on $a_1, a_2, \cdots, a_{n-1}$, but are independent of N and U.

are all units contained in F_N. The determinant

$$\Delta = \det(\log|\theta_N^{(j)} - a_i|), \qquad [1 \leqq j, i \leqq n-1],$$

is not 0 for sufficiently large N, as by (6) of Lemma 1 the non-diagonal terms are bounded, whereas the diagonal terms $\to -\infty$ for $N \to +\infty$. Hence $\theta_N^{(1)} - a_1, \theta_N^{(1)} - a_2, \cdots, \theta_N^{(1)} - a_{n-1}$ are multiplicatively independent in F_N. The same argument yields

$$(21) \qquad |\Delta| \leqq c_9(\log N)^{n-1}.$$

Now F_N is a totally real field of degree n over the rationals. So the regulator $R(F)$ is at most equal to the regulator of a set of $n-1$ independent units. Hence, by (21) $R(F_N) \leqq c_9(\log N)^{n-1}$. Take $\delta = \epsilon/2$ and choose ρ in Lemma 5 so that $\rho\delta > n-1$. As N satisfies Lemma 5, $|d(F_N)| > (\frac{1}{4}\log\frac{1}{2}N)^\rho$, so, for sufficiently large N,

$$R(F_N) \leqq c_9(\log N)^{n-1} \leqq |d(F_N)|^\delta.$$

Hence from (3), it follows that $h(F_N) > |d(F_N)|^{\frac{1}{2}-2\delta}$ which completes the proof of Theorem 1.

Section 4. We now carry over the proof to the case when F is not totally real. Let $n > 2, r_1 \geqq 0, r_2 \geqq 0$ be given integers such that $n = r_1 + 2r_2$. We prove

THEOREM 1*. *There exist infinitely many algebraic number fields K of degree n over the field P of rational numbers such that K has r_1 real conjugates and $2r_2$ non-real conjugates with the following property. The class-number h lies above $|d|^{\frac{1}{2}-\epsilon}$, d the discriminant, $\epsilon > 0$ a given constant.*

1. Let the a_λ, $(1 \leqq \lambda \leqq r_1)$, and the $a_\mu > 0$, $(r_1 + 1 \leqq \mu \leqq r+1)$, be distinct integers, where, as usual, $r = r_1 + r_2 - 1$. It can now be assumed that $r_2 > 0$, i. e. $r \geqq r_1$. We take $a_{r+1} = N$. Then define

$$(22) \qquad f(x) = \prod_{\lambda=1}^{r_1}(x - a_\lambda) \prod_{\mu=r_1+1}^{r+1}(x^2 + a_\mu) + 1.$$

so that, for $N \to +\infty$,

$$(23) \qquad f(x)/N \to \prod_{\lambda}(x - a_\lambda) \prod_{\mu \neq r+1}(x^2 + a_\mu).$$

We can denote r roots of $f(x)$ by $\theta_N^{(j)}$, $1 \leqq j \leqq r$, in such a fashion that

$$\theta_N^{(j)} \to a_j \quad (j = 1, 2, \cdots, r_1), \qquad \theta_N^{(j)} \to ia_j^{\frac{1}{2}} \quad (j = r_1 + 1, \cdots, r).$$

Then there exist constants $b_j \neq 0$ such that

(24) $N(\theta_N^{(j)} - a_j) \to b_j,\ (1 \leqq j \leqq r_1);\quad N(\theta_N^{(j)2} + a_j) \to b_j,\ (r_1 + 1 \leqq j \leqq r)$.

Actually,

$$b_j = -\prod_{\lambda \neq j} (a_j - a_\lambda)^{-1} \prod_{\mu \neq r+1} (a_j^2 + a_\mu)^{-1} \qquad (j \leqq r_1);$$

$$b_j = -\prod_{\lambda} (ia_j^{\frac{1}{2}} - a_\lambda)^{-1} \prod_{\mu \neq j, r+1} (-a_j + a_\mu)^{-1}, \qquad (r_1 + 1 \leqq j \leqq r).$$

For N sufficiently large, $\theta_N^{(j)}$, $1 \leqq j \leqq r_1$, is real and $\theta_N^{(j)}$, $r_1 + 1 \leqq j \leqq r_1 + r_2 - 1$, has positive imaginary part. Then $r_2 - 1$ further roots are obtained in the form $\bar{\theta}_N^{(j)}$, $(r_1 + 1 \leqq j \leqq r_1 + r_2 - 1)$. These have the limits $-ia_j^{\frac{1}{2}}$. Since the limit in (23) has degree $n - 2$, two of the roots of $f(x)$ tend to ∞. We see easily that these roots are not real for N sufficiently large and if $\theta_N^{(r+1)}$ is the one which has positive imaginary part, then $\theta_N^{(r+1)} = iN^{\frac{1}{2}} + O(1)$. (Actually one can show without difficulty that $\theta_N^{(r+1)} = iN^{\frac{1}{2}} + O(N^{-\frac{1}{2}(n-1)})$.

It follows from (22) that the $r_1 + r_2$ elements

$$(25) \qquad \theta - a_\lambda, \qquad \theta^2 + a_\mu$$

are units of the field $K = P(\theta)$, θ one of the θ_j.

2. We show that $f(x)$ is irreducible in P for sufficiently large N. If this was not so, let $f_0(x)$ be an irreducible factor in P such that $\theta_N^{(r+1)}$ is not a root of $f_0(x)$. Then $\bar{\theta}_N^{(r+1)}$ is not a root either. Let θ be a root of $f_0(x)$. All the conjugates of θ are close to fixed values. Take the unit $(\theta^2 + a_{r+1})^{-1}$. All its conjugates are less than 1 in absolute value for N sufficiently large. This is impossible. Thus $f(x)$ is irreducible in P. Hence $K = P(\theta)$ is a field of the given degree n with the given number r_1 of real conjugates.

3. Form the regulator R_0 of the r units (25) with $\mu \neq r + 1$ of the field $F_N = P(\theta)$, using the conjugates corresponding to $\theta_N^{(1)}, \cdots, \theta_N^{(r)}$. It follows from (24) that $|R_0| = O((\log N)^r)$. Hence $|R(F_N)| = O((\log N)^r)$. Now everything works as in the totally real case.

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THE REPRESENTATION OF INTEGERS BY CERTAIN RATIONAL FORMS.*

By E. G. Straus and J. D. Swift.

1. Introduction. In a previous paper [2] we discussed the representation of integers by $f(x,y) = (ax^2 + bxy + cy^2)/(p + qxy)$ where $a \mid (b,q)$ $c \mid (b,q)$. Similar questions have been discussed by other authors, [1], [3], [4]. In this paper we intend to analyze the underlying ideas and to extend their applications.

We wish to investigate an algebraic Diophantine equation in $n+1$ unknowns which is of degree no higher than the second in every unknown and of first degree in at least one of the unknowns. We distinguish the latter unknown by calling it z, and denote the other unknowns by $x_1, \cdots, x_n$ writing $x = (x_1, \cdots, x_n)$. Solving for z we obtain

$$(1) \qquad f(x) = z, \qquad f(x) = N(x)/D(x),$$

where $N(x)$, $D(x)$ are polynomials in x of degree no higher than 2 in each x_i. We shall be concerned here with the case $\deg D \geqq \deg N$.

Our main results are *finiteness results.* More precisely, we shall see that in certain cases there is a finite number of infinite classes of solutions of (1) which correspond to solutions of simpler Diophantine equations obtained from (1) by replacing some of the x_i by functions of the other x_j's. These we shall call the *regular* solutions of (1). The solutions which are not contained in the regular classes are called *exceptional.* They will be finite in number if certain divisibility conditions are satisfied. To a more limited extent we shall also be able to obtain *infinity results*; that is, prove in some cases that there is an infinity of exceptional solutions, if the divisibility conditions are violated.

The method of attack is a combination of two extremely simple ideas described in Sections 2, 3, 4. The remaining sections are devoted to a more complete discussion of special cases for the purpose of illustration. Finally, we shall discuss some possible extensions of our method.

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2. The critical cone. For the results in this section we need only the assumption $\deg D \geqq \deg N$ with no restriction on the degree in the individual x_i.

Definition. Let $D_1(x)$ be the homogeneous polynomial consisting of the terms of highest degree of $D(x)$. Then the *critical cone,* $\mathcal{C}$, of (1) is the locus

$$D_1(x) = 0. \tag{2}$$

A *conical neighborhood* of $\mathcal{C}$ is an open set of rays originating at the origin and containing $\mathcal{C}$.

LEMMA 1. *Let $\mathcal{K}$ be a conical neighborhood of $\mathcal{C}$, and let* $\deg D \geqq \deg N$. *Then* (1) *has at most a finite number of integral z for lattice points $x \notin \mathcal{K}$.*

Proof. Since $D_1(x) \neq 0$ in the exterior of $\mathcal{K}$ and since $\deg N \leqq \deg D$ there exist a radius r and a number M so that for $|x| = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}} > r$ and $x \notin \mathcal{K}$ we have $|f(x)| \leqq M$. On the other hand there is only a finite number of lattice points with $|x| \leqq r$.

COROLLARY. *If $D_1(x)$ is definite then $f(x)$ represents at most a finite number of integers.*

3. Conjugate points. For the results of this section we need only the assumption that N and D are at most quadratic in the x_i under consideration, with no restriction on their degrees.

Definition. If (1) is quadratic in x_i, then corresponding to each solution $(x_1, \cdots, x_i, \cdots, x_n, z)$ there is the *i-conjugate* solution $(x_1, \cdots, x'_i, \cdots, x_n, z)$, where x'_i is the conjugate of x_i when we consider (1) as an equation in x_i with all other unknowns fixed. We denote the i-conjugate of x by $x^{(i)}$.

A point x' is a conjugate of x if there exists a sequence $i_1, \cdots, i_k$ so that $x' = x^{(i_1)(i_2)\cdots(i_k)}$.

In general the conjugate of a lattice point has rational coordinates but is not a lattice point. However, writing

$$N(x) = a_i x_i^2 + b_i x_i + c_i, \qquad D(x) = A_i x_i^2 + B_i x_i + C_i,$$

where a_i, b_i, c_i, A_i, B_i, C_i are independent of x_i, we obtain from (1)

$$(a_i - zA_i)(x_i + x'_i) = zB_i - b_i. \tag{3}$$

This leads to the following result.

LEMMA 2. *The i-conjugate of a lattice point x is always a lattice point in either of the following cases:*

I. $b_i \equiv \alpha_i a_i$, $B_i \equiv \alpha_i A_i$ *(α_i-integer valued),*

II. $A_i \equiv 0$, $a_i \mid b_i$ *and* $a_i \mid B_i$,

where $a \mid b$ means that b/a is integer valued.

In case I we obtain

$$N(x) = a_i(x_i^2 + \alpha_i x_i) + c_i, \qquad D(x) = A_i(x_i^2 + \alpha_i x_i) + C_i.$$

Thus, if we set $u_i = x_i^2 + \alpha_i x_i$ then $f(x) = F(x_1, \cdots, u_i, \cdots, x_n)$ where F is rational. Thus in case I we consider instead the simpler Diophantine equation

$$(1') \qquad F(x_1, \cdots, u_i, \cdots, x_n) = z,$$

whose solutions obviously include all those obtained from (1) by setting $u_i = x_i^2 + \alpha_i x_i$.

In case II we may first consider the case $B_i \equiv 0$, that is, D independent of x_i. In this case we see that if $f(x)$ represents an integer at all, then it represents all the values of a certain quadratic polynomial, since we may replace x_i by $x_i + mD$ $(m = 0, \pm 1, \cdots)$ to obtain other integers.

In case $B_i \not\equiv 0$ we can write

$$(4) \qquad B_i^2 f = a_i B_i x_i - a_i C_i + B_i b_i + (a_i C_i^2 - b_i B_i C_i + c_i B_i^2)/D.$$

Writing (4) for $x^{(i)}$, subtracting and dividing by $x_i - x'_i$ we obtain

$$(5) \qquad a_i D(x) D(x^{(i)}) = a_i C_i^2 - b_i B_i C_i + c_i B_i^2.$$

4. The method. We can now combine the methods of §§ 2, 3 to state the following.

FINITENESS RESULT. *If every lattice point x satisfying* (1) *has a conjugate lattice point in the exterior of some conical neighborhood of the critical cone, then* (1) *has solutions for at most a finite number of z.*

It will be somewhat more difficult to state a general description of the infinity results and we shall therefore do this by example.

5. The case $n = 2$. We set $x_1 = x$, $x_2 = y$ and have

$$(1'') \qquad f(x, y) = N(x, y)/D(x, y) = z.$$

We first consider the case $\deg D = 4$. That is, $N = ax^2y^2 + \cdots$, $D = Ax^2y^2 + \cdots$, $(A \neq 0)$, where $\cdots$ stands for terms of lower degree. Hence $Af(x,y) = a + N^*/D$ where $\deg N^* \leqq 3$. Thus there exist numbers x_0, y_0, M so that if $x \geqq x_0$ and $y \geqq y_0$ then $f(x,y) \leqq M$. We may therefore restrict our attention to the strip neighborhood $|x| < x_0$ or $|y| < y_0$ of the critical cone $xy = 0$.

This reduces the problem to the consideration of a finite number of rational functions of *one* variable

$$g_x(y) = f(x,y), \quad x = 0, \pm 1, \cdots, \pm(x_0 - 1);$$
$$h_y(x) = f(x,y), \quad y = 0, \pm 1, \cdots, \pm(y_0 - 1).$$

The *regular* values of x are those (if any) for which $g_x(y)$ is a polynomial in y; similarly the regular values of y are those for which $h_y(x)$ is a polynomial in x. There are at most 2 regular values for x and for y. If there are such regular values then $g_x(y)$ and $h_y(x)$ may represent an infinity of regular integers. For all other values of (x,y) we obtain at most a finite number of exceptional integers.

We next consider the case $\deg D = 3$. That is, $N = xy(ax + by) + \cdots$, $D = xy(Ax + By) + \cdots$. The method first used can be used with minor modification in the special cases (i) $\deg N = 1$ or (ii) $a/A = b/B$, or (iii) neither N or D contains quadratic terms. Since it involves few new ideas we shall not elaborate on it.

Outside the above cases we shall not be able to proceed without the conjugate point method. If case I of Lemma 2 is satisfied for either x or y then we have seen that by a change of variable we reduce the degree of D to 2, to be discussed below.

For case II of Lemma 2, D must be linear in one of the variables, say y, so that $D = Ax^2y + \cdots$. Hence we may write $Af = a + N^*/D$, $N^* = b^*xy^2 + \cdots$.

The lattice points (x,y) for which $N^* = 0$ lead to at most one integer $z = a/A$. If $N^* \neq 0$ then writing $D = (A_1x^2 + B_1x + C_1)y + A_2x^2 + B_2x + C_2$ we have either $y = 0$ or $A_1y + A_2 = 0$ or there exists an M_1 such that $|x/y| \leqq M_1$. If $y = -A_2/A_1$ is an integer then this special value leads to easily determined solutions which may form a regular class or be finite in number. Unless $A_2 = 0$ the value $y = 0$ leads at most to a finite number of exceptional solutions.

Writing now $N^* = a(x)y^2 + b(x)y + c(x)$, $D = B(x)y + C(x)$ we have according to (5)

$$(5') \qquad a(x)(By + C)(By' + C) = aC^2 - bBC + cB^2.$$

The case $a \equiv 0$ can be treated with the method used for $\deg D = 4$. If $a \neq 0$ then the special values of x for which $x = 0$ or $a(x) = 0$ or $B(x) = 0$ lead to easily determined solutions which may contain regular classes. Finally, if the divisibility conditions $a \mid b$ and $a \mid B$ of Lemma 2 are satisfied and $xaB \neq 0$ then we obtain from (5′)

$$\frac{y}{x}\cdot\frac{y'}{x} + \frac{C}{xB}\left(\frac{y}{x} + \frac{y'}{x}\right) + \left(\frac{bC}{x^2aB} - \frac{c}{x^2a}\right) = 0 \tag{6}$$

$$= \frac{y}{x}\cdot\frac{y'}{x} + \alpha\left(\frac{y}{x} + \frac{y'}{x}\right) + \beta.$$

The degrees of the denominators of α and β are no less than the degrees of their numerators. Hence α, β are bounded. Thus, if we choose $|y| \leqq |y'|$ then there exists an M_2 so that $|y/x| \leqq M_2$.

To sum up. We may obtain regular solutions or a finite number of solutions for $y = -A_2/A_1$, and the values of x for which $x = 0$ or $a(x) = 0$ or $B(x) = 0$. In addition there may be a finite number of exceptional values for $y = 0$. All other lattice points have a conjugate for which both $|x/y| \leqq M_1$ and $|y/x| \leqq M_2$, that is, lying in the exterior of a conical neighborhood of the critical cone $xy = 0$. According to Lemma 1 this leads to at most a finite number of exceptional values of $f(x, y)$.

Finally we consider the case $\deg D = 2$. Let $D_1(x, y) = Ax^2 + Bxy + Cy^2$ be the quadratic part of D and $\Delta = B^2 - 4AC$.

The case $\Delta < 0$, that is D_1 definite, is covered by Lemma 1.

In case $\Delta = 0$ we can make a unimodular transformation so that $D = Ax^2 + B_1x + B_2y + C$. Here there may well be an infinite number of exceptional values. For example if the congruences

$$Ax^2 + B_1x + C \equiv \pm 1 \pmod{B_2} \tag{7}$$

have solutions, then for each x satisfying (7) there is a y so that $D(x, y) = \pm 1$ and hence certainly $f(x, y) = z$ is an integer. From the conjugate point method we obtain only the existence of constants M_1, M_2 so that the regular integers are represented by $x = 0$ while all but a finite number of exceptional integers are represented by lattice points satisfying $M_1 |x| \leqq |y| \leqq M_2x^2$.

The case $\Delta > 0$, $\Delta \neq$ square, is similar to the preceding one. The equations $D = \pm 1$ may again have infinitely many solutions.

In case $\Delta > 0$, $\Delta =$ square, we can write $D = A(\alpha x + \beta y)(\gamma x + \delta y) + B_1x + B_2y + C$ where $(\alpha, \beta) = (\gamma, \delta) = 1$, $\alpha\delta - \beta\gamma \neq 0$. If we make the transformation $u = \alpha x + \beta y$, $v = \gamma x + \delta y$, then every lattice point (x, y) there corresponds a lattice point (u, v). The converse is not true unless

$\alpha\delta - \beta\gamma = \pm 1$, but for finiteness results we do not need it. We can thus restrict our attention to the case

$$D(x,y) = Axy + B_1x + B_2y + C.$$

If we write $N(x,y) = a_1x^2 + a_2xy + a_3y^2 + b_1x + b_2y + c$, then the divisibility conditions of case II, Lemma 2 become $a_1 \mid a_2$, $a_1 \mid A$, $a_1 \mid B_2$ and $a_3 \mid a_2$, $a_3 \mid A$, $a_3 \mid B_1$ respectively. If both sets of divisibility conditions are satisfied, then from (5) we obtain

$$(8) \quad a_3(Axy + B_1x + B_2y + C)(Axy' + B_1x + B_2y' + C)$$
$$= a_3C^2 - (a_2x + b_2)(Ax + B_2)C + (a_1x^2 + b_1x + c)(Ax + B_2)^2$$

and

$$(9) \quad a_1(Axy + B_1x + B_2y + C)(Ax'y + B_1x' + B_2y + C)$$
$$= a_1C^2 - (a_2y + b_1)(Ay + B_1)C + (a_3y^2 + b_2y + c)(Ay + B_1)^2.$$

From (8) we see that either $x = 0$ or $x = -B_2/A$ or there exists an M_1 such that if we choose $|y| \leqq |y'|$ then $|y/x| \leqq M_1$. Similarly we obtain from (9) that either $y = 0$ or $y = -B_1/A$ or there exists an M_2 such that if we choose $|x| \leqq |x'|$ then $|x/y| \leqq M_2$. Regular values may be represented by $x = -B_2/A$ or $y = -B_1/A$; the values $x = 0$ or $y = 0$ give rise to only a finite number of exceptional values unless they happen to coincide with the regular values. All lattice points (x,y) which are not conjugate to one of the above points have a conjugate—obtained by choosing x minimal and y minimal for that x—which lies in the exterior of a conical neighborhood of the critical cone $xy = 0$. Hence there is only a finite number of exceptional values of z.

6. An example $n = 3$. We consider the equation

$$f(x,y,z) = (x^2 + y^2 + z^2)/(xyz + 1) = u.$$

This is the analogue of [2, Example 1]. The divisibility conditions of Lemma 2, case II are satisfied for x, y, z; and (5) becomes (for the variable z)

$$(xyz + 1)(xyz' + 1) = 1 + x^2y^2(x^2 + y^2).$$

Hence either $xy = 0$ or, if $|z| \leqq |z'|$ then

$$(10) \qquad (xyz + 1)^2 \leqq 1 + x^2y^2(x^2 + y^2).$$

We can obtain the analogous inequality for the other two variables.

Thus the regular values of $f(x,y,z)$ are the ones obtained on the critical cone $xyz=0$; that is, the numbers which are the sums of two squares.

If $xyz\neq 0$ we may restrict our attention to the points (x,y,z) for which $0<|x|\leqq y\leqq z$. In case $xyz>0$ equation (10) yields $z^2<x^2+y^2$ and hence $y^2>z^2/2$. Thus $f(x,y,z)<3z^2/2^{-\frac{1}{2}}xz^2=3\cdot 2^{\frac{1}{2}}/x$. The only possible positive values for u which can be represented by exceptional lattice points are therefore 1, 2, 3, 4. It is easily seen that the equations $f(1,y,z)=1$, $f(2,y,z)=1$, $f(3,y,z)=1$, $f(4,y,z)=1$ have no solutions with $yz\neq 0$. The equations $f(1,y,z)=2$, $f(2,y,z)=2$ do have solutions with $yz\neq 0$; however these solutions can be seen to be conjugates of solutions with $yz=0$. Finally $f(1,y,z)=3$ and $f(1,y,z)=4$ have no solutions.

We have thus seen that there are no positive exceptional values of u, and that the positive regular values of u are represented only by the conjugates of regular lattice points.

By inspection we obtain the negative exceptional values $f(-1,2,2)=-3$, $f(-1,1,2)=-6$. From (10) we obtain for $xyz<0$

$$x^2y^2(z-1)^2\leqq x^2y^2(1+x^2+y^2) \text{ or } 3y^2+1\geqq(z-1)^2.$$

Hence

$$f(x,y,z)\leqq 3z^2/[|x|z(z-1)/3^{\frac{1}{2}}-1]\leqq(3\cdot 3^{\frac{1}{2}}/|x|)\cdot z/(z-2).$$

We see that for $z\geqq 8$ this leads to $|f|\leqq 6$. Thus we need inspect only a finite number of cases to exclude exceptional values less than -6. For $z\geqq 8$ the values -4, -5 are possible only for $x=-1$ but the equations $f(-1,y,z)=-4$, $f(-1,y,z)=-5$ have no solutions. To exclude $u=-2$ we observe that $f(-1,y,z)=-2$, $f(-2,y,z)=-2$ are both impossible. The impossibility of $u=-1$ follows from the impossibility of the congruence $x^2+y^2+z^2+xyz\equiv -1 \pmod 4$.

7. Infinity results. In [2] we discussed the case

$$f(x,y)=(x^2+2y^2)/(1+xy)$$

which violates the divisibility condition of Lemma 2, case II for y. We can now argue that there must exist an infinity of exceptional values as follows.

We have $x'=-x+yz$, $y'=-y+\frac{1}{2}xz$. For even z this leads to the conjugate point method and our previously discussed finiteness result. For odd z we see that a non-lattice point

$$x=\xi\cdot 2^{-k},\qquad y=\eta\cdot 2^{-k-\epsilon}\qquad (\xi,\eta \text{ odd}; \epsilon=0,1)$$

has a conjugate which is a lattice point. Hence every integer represented by

$$f_k(x,y) = (x^2+2y^2)/(2^k+xy)$$

for odd x, y is also represented by $f(x,y)$. But every f_k represents at least four exceptional integers for odd x, y; namely those obtained from $xy = -2^k \pm 1$. It is clear that these values for f_k are unbounded with k. Hence $f(x,y)$ represents infinitely many odd exceptional numbers. It is easy to apply this process to other cases, but it may be difficult to formulate a general theorem.

8. Conclusion. Our method is clearly not restricted to the classes of equations which we have considered here. Changes in variable may bring a rational function into the form we considered.

A more interesting possibility is that of increasing the number of variables in order to decrease the degree of the equation in each variable and then restrict attention to the case in which the new variables are functions of the old variables.

As an example of this last possibility we discuss the equation

$$u = (x^4+x^2+y^2)/(x^3y+1). \tag{11}$$

The substitution $x^2 = z$ brings this to the form

$$u = (x^2+y^2+z^2)/(xyz+1) \tag{12}$$

which we discussed in Section 6. We see therefore immediately that u must be either the sum of two squares or one of the exceptional values -3, -6. For $x = -1$, $y = 2$ we obtain $u = -6$, while $x = 2$, $y = -1$ yields $u = -3$. Now for the regular values of u we know that they must be represented by a lattice point (x,y,z) so that $z = x^2$ and so that there is a conjugate of one of the forms $(0,a,b)$, $(a,0,b)$, $(a,b,0)$. From (5) we see that

$$x' = -x + uyz, \qquad y' = -y + uxz, \qquad z' = -z + uxy.$$

Thus, if (x_1, y_1, z_1) is a conjugate of (x,y,z), then $x_1 \equiv \pm x$, $y_1 \equiv \pm y$, $z_1 \equiv \pm z \pmod{u}$. Hence if (x,y,x^2) is conjugate to $(0,a,b)$ then $x \equiv x^2 \equiv 0$ $(\mathrm{mod}\ (a^2+b^2))$ and $b \equiv \pm x^2 \equiv 0$ $(\mathrm{mod}\ (a^2+b^2))$ which implies $b = 0$. The regular values thus obtained are therefore exactly those obtained from (11) by setting $x = 0$, that is the squares.

If (x,y,x^2) is conjugate to $(a,0,b)$ then, since $y \equiv 0$ $(\mathrm{mod}\ (a^2+b^2))$, at every stage, $x \equiv \pm a,\ x^2 \equiv \pm b$ $(\mathrm{mod}\ (a^2+b^2)^2)$ and hence $b^2 \equiv a^4$ $(\mathrm{mod}\ (a^2+b^2)^2)$, but $b^2 - a^4 < (a^2+b^2)^2$. Hence $b^2 = a^4$ and $u = a^2 + a^4$

is the regular value obtained from (11) by setting $y=0$. Finally, if (x,y,x^2) is conjugate to $(a,b,0)$ then

$$x \equiv \pm a \ (\mathrm{mod}(a^2+b^2)),\ x^2 \equiv a^2 \equiv 0 \ (\mathrm{mod}\ (a^2+b^2)).$$

This is possible only if $b=0$ which we have discussed before.

The conjugate point idea is, of course, not restricted to equations which are of second degree in the unknowns, however, our divisibility conditions of Lemma 2 will then have to be replaced by more complicated and cumbersome conditions.

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IDEALS AND POLYNOMIAL FUNCTIONS.*[1]

By D. J. Lewis.

1. Introduction. Let K be either an algebraic number field or a function field over a finite field or any completion of such fields under a rank one, non-archimedean valuation. Let $\mathfrak{O}$ be the integrally closed ring of integers of K, $\mathfrak{p}$ any prime ideal of $\mathfrak{O}$, and π any element of $\mathfrak{p}$ not in $\mathfrak{p}^2$. Then $\mathfrak{O}/\mathfrak{p}$ is isomorphic to a finite field $GF(q)$, where $q = p^f$, p a rational prime. Let $B_{\mathfrak{p}^m} = B_m$ be the set of polynomials in $\mathfrak{O}[x]$, which when considered as functions on $\mathfrak{O}$ map $\mathfrak{O}$ into $\mathfrak{p}^m$; i. e.,

$$B_m = \{f(x) \text{ in } \mathfrak{O}[x] \text{ such that } f: \mathfrak{O} \rightarrow \mathfrak{p}^m\}.$$

Clearly B_m is an ideal in $\mathfrak{O}[x]$.

It is well known [1] that $B_1 = (x^q - x, \pi)$, in fact this result has now become a part of elementary algebra and number theory [2]. Here we analyze B_m, when $m \neq 1$. Clearly $B_0 = \mathfrak{O}[x]$, and $B_m \supset B_{m+1}$. We show that B_m is generated by $m+1$ elements, and give a specific set of generators. The proof is by induction and is of an elementary nature. The results may also be viewed as a concrete realization of a general theory of rings, see [3]; results for the case of polynomials of several indeterminates are also obtained. Because of the computations involved, we first consider the case of one indeterminate and then outline the steps necessary for the case of several indeterminates. These results have been applied to a study of Diophantine equations, which will appear in another paper.

2. Preliminaries. Define

$$\tau_0(x) = x, \quad \tau_{n+1}(x) = \tau_n{}^q(x) - \pi^{q^n-1}\tau_n(x) \quad \text{for } n \geqq 0.$$

Let $Q(n) = (q^n - 1)/(q-1)$. It is easily verified that $\tau_n(x)$ is in $B_{Q(n)}$.

The expression of a rational positive integer m as

$$m = \mu_1 + \mu_2 Q(2) + \cdot\cdot\cdot + \mu_t Q(t),$$

where $0 \leqq \mu_i \leqq q$, $\mu_t \neq 0$, and $\mu_i = 0$ for $1 \leqq i < j$ if $\mu_j = q$, is unique. We use it to define the following polynomials:

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[1] This paper is a part of the results presented to the American Mathematical Society, August 31, 1953, under the title: "Polynomial functions over the residue ring $\mathfrak{O}/\mathfrak{p}^m$." This work was done while the author was a National Science Foundation Fellow at the Institute for Advanced Study.

$$\lambda_0 = 1, \ \lambda_m = \tau_1^{\mu_1}\tau_2^{\mu_2}\cdots\tau_t^{\mu_t} \text{ if } m > 0.$$

Clearly λ_m is an element of B_m.

Let $A_m = (\lambda_m, \pi\lambda_{m-1}, \cdots, \pi^{m-1}\lambda_1, \pi^m)$, $m \geqq 1$. Then $A_m \subset B_m$. We contend that $B_m = A_m$. As we shall sometime make a change in variable, for clarity we may sometimes write $A_m(x)$ or $B_m(x)$ to indicate we are operating in $\mathfrak{O}[x]$.

We make several observations concerning A_m.

THEOREM I. i) $A_m \supset A_{m+1}$.

ii) $A_m \cdot A_n \subset A_{m+n}$.

iii) *If $g(z)$ is in $\mathfrak{O}[x]$ and $f(x)$ is in $A_m(x)$, then $f(g(z))$ is in $A_m(z)$; symbolically $A_m(g(z)) \subset A_m(z)$.*

Proof. In view of the definition of A_m, (i) is evident, provided λ_{m+1} is in A_m. Consider the expansion

$$m + 1 = \nu_1 + \nu_2 Q(2) + \cdots + \nu_t Q(t).$$

If $\nu_1 \neq 0$, $\lambda_{m+1} = \tau_1\lambda_m$ and hence is in A_m. If $\nu_1 = \nu_2 = \cdots = \nu_r = 0$, and $\nu_{r+1} \neq 0$, then $\lambda_{m+1} = \lambda_m - \pi^{q^r-1}\lambda_{(m-q^r+1)}$, which is in A_m.

By definition τ_n^{α} is in $A_{\alpha Q(n)}$ if $0 \leqq \alpha \leqq q$, but because of (i) we have this for all $\alpha \geqq 0$. Then by induction $\tau_n^{\beta}\lambda_m$ is in $A_{m+\beta Q(n)}$ if $q \geqq \beta \geqq 0$, and finally $\pi^s\lambda_m\lambda_n$ is in A_{s+m+n}. Thus proving (ii).

Using (ii) and the definition of τ_i and λ_s, we obtain $\lambda_s^q \equiv \pi^{qs-s}\lambda_s \pmod{A_{qs+1}}$.

Observe that the binomial coefficients $C_i^q \equiv 0 \pmod{\mathfrak{p}}$ if $1 \leqq i \leqq q-1$, also that $\tau_1(z^r) = \sum_{i=1}^{n} z^{(r-i)q+i-1}\tau_1(z)$. It follows immediately that $\tau_1(g(z))$ is in $A_1(z)$. Because of (ii) it is clearly sufficient for proving (iii) to show that $\tau_n(g(z))$ is in $A_{Q(n)}(z)$. This is proved by induction:

Suppose $\tau_k(g(z))$ is in $A_{Q(k)}(z)$ if $1 \leqq k \leqq n$. Then:

$$\begin{aligned}
\tau_{n+1}(g(z)) &= \Big[\sum_{s=0}^{Q(n)} h_s(z)\pi^{Q(n)-s}\lambda_s(z)\Big]^q - \pi^{q^n-1}\sum_{s=0}^{Q(n)} h_s(z)\pi^{Q(n)-s}\lambda_s(z)\\
&\equiv \sum_{s=0}^{Q(n)} \big[h_s(z^q)\pi^{qQ(n)-qs}\lambda_s^q(z) - h_s(z)\pi^{Q(n+1)-s-1}\lambda_s(z)\big] \pmod{A_{Q(n+1)}}\\
&\equiv \sum_{s=0}^{Q(n)} \big[h_s(z^q)\pi^{qQ(n)-s}\lambda_s(z) - h_s(z)\pi^{qQ(n)-s}\lambda_s(z)\big] \pmod{A_{Q(n+1)}}\\
&\equiv \sum_{s=0}^{Q(n)} \pi^{qQ(n)-s}\lambda_s(z)\big[h_s(z^q) - h_s(z)\big] \pmod{A_{Q(n+1)}}\\
&\equiv \sum_{s=0}^{Q(n)} \pi^{qQ(n)-s}\lambda_s(z)\tau_1(h(z)) \equiv 0 \pmod{A_{Q(n+1)}}.
\end{aligned}$$

If $m=\mu_1+\mu_2 Q(2)+\cdots+\mu_t Q(t)$, define

$$m^*=\mu_2+\mu_3 Q(2)+\cdots+\mu_t Q(t-1).$$

Then $m-m^*=\sum_{i=1}^{t}\mu_i q^{i-1}$ and $m\leqq q$ implies $m^*=0$.

We shall have use of the following lemma.

LEMMA 1. *For every a in $\mathfrak{O}$, there exists a c in $\mathfrak{O}$ such that $a\equiv c$ $(\operatorname{mod}\mathfrak{p})$ and such that for every λ_s there is a polynomial $h_s(z)$ in B_{s^*-1} such that*

$$\lambda_s(c-\pi z)=\pi^{s-s^*}z^{\mu_1(s)}\lambda_{s^*}(z)+\pi^{s-s^*+1}h_s(z).$$

Proof. Take $c=a+\tau_1(a)$. Then $\tau_1(c)\equiv 0$ $(\operatorname{mod}\mathfrak{p}^2)$ and consequently $\tau_1(c-\pi z)=\pi z+\pi^2 h_1(z)$. By induction we show that

$$\tau_{n+1}(c-\pi z)\equiv\pi^{q^n}\tau_n(z) \qquad (\operatorname{mod}\mathfrak{p}^{q^n+1}),$$

the congruence being coefficient-wise. Suppose such is the case for τ_k, where $1\leqq k\leqq n$. Then

$$\begin{aligned}\tau_{n+1}(c-\pi z)&=\tau_n{}^q(c-\pi z)-\pi^{q^n-1}\tau_n(c-\pi z)\\&=\pi^{q^n}[\tau_{n-1}(z)+\pi k(z)]^q-\pi^{q^n+q^{n-1}-1}[\tau_{n-1}(z)+\pi k(z)]\\&\equiv\pi^{q^n}[\tau_{n-1}{}^q(z)-\pi^{q^{n-1}-1}\tau_{n-1}(z)]\ \ (\operatorname{mod}\mathfrak{p}^{q^n+1})\\&\equiv\pi^{q^n}\tau_n(z)\ \ (\operatorname{mod}\mathfrak{p}^{q^n+1}).\end{aligned}$$

If $s=\mu_1(s)+\mu_2(s)Q(2)+\cdots+\mu_t(s)Q(t)$, we obtain

$$\begin{aligned}\lambda_s(c-\pi z)&=\prod_{i=1}^{t}\tau_i{}^{\mu_i(s)}(c-\pi z)=\pi^{s-s^*}\prod_{i=1}^{t}[\tau_{i-1}(z)+\pi k_i(z)]^{\mu_i(s)}\\&=\pi^{s-s^*}z^{\mu_1(s)}\lambda_{s^*}(z)+\pi^{s-s^*+1}h_s(z).\end{aligned}$$

Since λ_s and $\pi^{s-s^*}\lambda_{s^*}$ are in B_s, it follows that h_s is in B_{s^*-1}.

3. Proof of the contention that $B_m=A_m$. Let

$$\Gamma=\{\lambda_m \text{ such that } m=\mu_1+\mu_2 Q(2)+\cdots+\mu_t Q(t);\ 0\leqq\mu_i<q\}.$$

Then for each non-negative integer r, Γ contains a unique polynomial of degree rq.

Let $f(x)$ be any polynomial in $\mathfrak{O}[x]$. Let d denote the degree of $f(x)$ and define e such that $eq\leqq d<q(e+1)$. Then $f(x)$ can be expressed uniquely as $f(x)=g(x)\lambda(x)+f^*(x)$, where the degree of $g(x)$ is less

than q, $\lambda(x)$ is the unique polynomial in Γ of degree eq and the degree of $f^*(x)$ is less than eq.

If c is the largest power of π dividing any coefficient of $g(x)$ we may write $g(x)=\sum_{i=0}^{c}\pi^i g_i(x)$, where the non-zero coefficients of the $g_i(x)$ are not divisible by π. Clearly the $g_i(x)$ are uniquely determined by $g(x)$, hence by $f(x)$. Continuing the process on the residual polynomial we arrive at a unique expression for $f(x)$ of the form.

$$f(x)=\sum_{\Gamma}\sum_{t} g_{k,t}(x)\pi^t\lambda_k(x), \tag{1}$$

where the $\lambda_k(x)$ are in Γ, the degree of each $g_{k,t}(x)$ is less than q, the non-zero coefficients of the $g_{k,t}(x)$ are not in $\mathfrak{p}$; and almost all of the $g_{k,t}(x)$ are the zero polynomial.

As previously remarked $B_1=A_1$. Assume $B_i=A_i$, for $1\leqq i\leqq m$ and suppose $f(x)$ is in B_{m+1}. Now $B_{m+1}\subset B_m=A_m$, hence using (1) we obtain

$$f(x)\equiv\sum_{s=0}^{m}{}' g_s(x)\pi^{m-s}\lambda_s(x) \qquad (\operatorname{mod} A_{m+1}) \tag{2}$$

where the $g_s(x)$ are of degree less than q and their non-zero coefficients are not in $\mathfrak{p}$, and where the accent mark indicates that the sum ranges over the λ_s, $0\leqq s\leqq m$, which are in Γ.

If $s\leqq m$, we have $s^*<m$, then in light of the induction hypothesis for $s\leqq m$, the $h_s(z)$ of Lemma 1 is in A_{s^*-1}. Hence

$$\begin{aligned} f(c-\pi z) &\equiv \sum_{s=0}^{m}{}' \,(g_s(a)+\pi w_s(z))\pi^{m-s}\lambda_s(c-\pi z) \quad (\operatorname{mod} A_{m+1}) \\ &\equiv \sum_{s=0}^{m}{}' g_s(a)\,[z^{\mu_1(s)}\pi^{m-s^*}\lambda_{s^*}(z)+\pi^{m-s^*+1}h_s(z)] \quad (\operatorname{mod} A_{m+1}) \\ &\equiv [\textstyle\sum'' \gamma_s z^{\mu_1(s)}]\pi^{m-m^*}\lambda_{m^*}(z)+\pi^{m-m^*+1}R(z) \quad (\operatorname{mod} A_{m+1}) \end{aligned}$$

where $R(z)$ is a polynomial in A_{m^*-1}; where the double accent indicates that the sum ranges over those s for which λ_s is in Γ and for which $s^*=m^*$; and where $\gamma_s=g_s(a)$ if $g_s(a)$ not in $\mathfrak{p}$ and $\gamma_s=0$ if $g_s(a)$ is in $\mathfrak{p}$.

If $s^*=m^*=t^*$ and $s\neq t$, then $\mu_1(s)\neq\mu_1(t)$, thus the γ_s are coefficients of different powers of z. Since $\mu_1(s)<q$ for all s for which λ_s is in Γ, $G(z)=\sum''\gamma_s z^{\mu_1(s)}$ is a g-polynomial of the type specified in (1).

Let $H(z)=G(z)\lambda_{m^*}(z)+\pi R(z)$, then $f(c-\pi z)\equiv\pi^{m-m^*}H(z)$ (mod A_{m+1}). By assumption $f(x)$ is in B_{m+1}, hence $H(z)$ is in B_{m^*+1}. But $m^*+1\leqq m$, thus by the induction hypothesis $B_{m^*+1}=A_{m^*+1}$. Hence, $H(z)$ is in A_{m^*+1},

consequently $G(z)$ is the zero polynomial and we have $g_s(a) \equiv 0 \pmod p$, for all s for which λ_s is in Γ and for which $s^* = m^*$.

Since this computation is true for every a in $\mathfrak{O}$, we have that for these s, $g_s(x)$ is in $B_1(x) = A_1(x)$ and hence are the zero-polynomial. In particular if λ_m is in Γ, $g_m(x)$ is the zero polynomial. Using this fact and (2), we obtain

$$f(x) \equiv \pi[\sum_{s=0}^{m-1}{}' g_s(x)\pi^{m-1-s}\lambda_s(x)] = \pi F(x) \pmod{A_{m+1}}.$$

But then $F(x)$ is in $B_m = A_m$ and so must be the zero polynomial. Thus $f(x)$ is in A_{m+1}. Since $f(x)$ was any polynomial from B_{m+1}, we have proved that $B_{m+1} = A_{m+1}$, proving

THEOREM II. *$A_m = B_m$, for all $m \geqq 1$.*

We also obtain that $B_\infty = \bigcap_{m \geqq 1} B_m = 0$. For suppose $f(x)$ is in B_∞, let d denote the degree of $f(x)$ and let r be the highest power of π dividing all of the coefficients of $f(x)$. Then $f(x)$ is not in B_{d+r+1}, hence not in B_∞.

4. The case of more than one indeterminant. Where convenient we shall denote a polynomial of $\mathfrak{O}[x_1, x_2, \cdots, x_n]$ by $f(X)$ and sometimes just by f. Let $\mathfrak{B}$ be the n-dimensional vector space over $\mathfrak{O}$. Let

$$\mathfrak{B}_m = \{f \text{ in } \mathfrak{O}[x_1, x_2, \cdots, x_n] \text{ such that } f: \mathfrak{B} \to \mathfrak{p}^m\}.$$

It is well known [1] that $\mathfrak{B}_1 = (\tau_1(x_1), \tau_1(x_2), \cdots, \tau_1(x_n), \pi)$.

Consider all partitions, $(\rho) = (\rho_1, \rho_2, \rho_3, \cdots, \rho_n)$ of m into n non-negative integers; i.e., $m = \rho_1 + \rho_2 + \cdots + \rho_n$, where $\rho_i \geqq 0$. Define

$$\Lambda_m^{(\rho)}(X) = \prod_{i=1}^{n} \lambda_{\rho_i}(x_i).$$

Clearly each $\Lambda_m^{(\rho)}$ is in $\mathfrak{B}_m$. For completeness we shall outline the proof of the following result:

THEOREM III. *$\mathfrak{B}_m = (\pi\mathfrak{B}_{m-1}, \Lambda_m^{(\rho)})$, where (ρ) ranges over all partitions of m.*

Let $\mathfrak{A}_m = (\pi\mathfrak{A}_{m-1}, \Lambda_m^{(\rho)})$, where (ρ) ranges over all partitions of m. Then we obtain

LEMMA 2. i) $\mathfrak{A}_m \supset \mathfrak{A}_{m+1}$

ii) $\mathfrak{A}_m \cdot \mathfrak{A}_n \subset \mathfrak{A}_{m+n}$

iii) *If $g_i(Z)$ are in $\mathfrak{O}[z_1, z_2, \cdots, z_n]$ and $f(X)$ is in $\mathfrak{A}_m(X)$, then $f(g_1(Z), g_2(Z), \cdots, g_n(Z))$ is in $\mathfrak{A}_m(Z)$.*

If $(\rho) = (\rho_1, \rho_2, \cdots, \rho_n)$ define $\rho^* = \sum \rho_i^*$ where ρ_i^* is defined as in Section 2. Then $\rho^* < m$ and $(\rho^*) = (\rho_1^*, \rho_2^*, \cdots, \rho_n^*)$ is a partition of ρ^*. Using Lemmas 1, 2 and definitions, we obtain

LEMMA 3. *For every vector* $\mathfrak{a} = (a_1, a_2, \cdots, a_n)$ *in* $\mathfrak{B}$ *there exists a vector* $\mathfrak{c} = (c_1, c_2, \cdots, c_n)$ *in* $\mathfrak{B}$ *such that* $a_i \equiv c_i \pmod{\mathfrak{p}}$ *for* $1 \leqq i \leqq n$, *and such that for every* $\Lambda_s^{(\sigma)}$ *there is a polynomial* $H^{(\sigma)}$ *in* $\mathfrak{B}_{\sigma^*-1}$ *such that*
$\Lambda_s^{(\sigma)}(\mathfrak{c} - \pi Z) = \pi^{s-\sigma^*} k^{(\sigma)}(Z) \Lambda_{\sigma^*}{}^{(\sigma^*)}(Z) + \pi^{s-\sigma^*+1} H^{(\sigma)}$, *where* $k^{(\sigma)}(Z) = \prod_{i=1}^{n} z_i^{\mu_1(\sigma_i)}$.

Let Γ_0 be the subset of the $\Lambda_m^{(\rho)}$ which do not have a factor $\tau_i(x_j)$, appearing to the q-th power. We can develop an algorithm which leads to a unique expression for each f in $\mathfrak{O}[x_1, x_2, \cdots, x_n]$ in the form

$$f(X) = \sum_{\Gamma_0} \sum_{t} g_{s,(\sigma),t}(X) \pi^t \Lambda_s^{(\sigma)}(X) \tag{3}$$

where the $\Lambda_s^{(\sigma)}$ are in Γ_0, the degree of each indeterminate in $g_{s,(\sigma),t}$ is less than q, the coefficients of the $g_{s,(\sigma),t}$ are either zero or not in $\mathfrak{p}$, and almost all of the $g_{s,(\sigma),t}$ are the zero polynomial.

As noted $\mathfrak{B}_1 = \mathfrak{A}_1$, we suppose $\mathfrak{B}_i = \mathfrak{A}_i$ for $i \leq m$ and prove $\mathfrak{B}_{m+1} = \mathfrak{A}_{m+1}$. Let f be in $\mathfrak{B}_{m+1} \subset \mathfrak{B}_m = \mathfrak{A}_m$. Then

$$f(X) = \sum_{s=0}^{m} \sum_{(\sigma)}{}' \pi^{m-s} g_{s,(\sigma)}(X) \Lambda_s^{(\sigma)}(X) \pmod{\mathfrak{A}_{m+1}}$$

where the accent indicates the sum ranges over that $\Lambda_s^{(\sigma)}$ in Γ_0, and the $g_{s,(\sigma)}$ are as in (3).

Let $u = \mathrm{Max}\{\sigma^*$, where (σ) is a partition of s and $s \leqq m\}$. Then

$$f(X) = \sum_{r=0}^{u} f_r(X) \pmod{\mathfrak{A}_{m+1}}, \text{ where } f_r(X) = \sum_{(\sigma)}^{\sigma^*=r}{}' \pi^{m-s} g_{s,(\sigma)}(X) \Lambda_s^{(\sigma)}(X).$$

Here the sum ranges over those integers s and their partitions (σ) for which $\Lambda_s^{(\sigma)}$ is in Γ_0, $s \leqq m$ and $\sigma^* = r$. Then

$$f(\mathfrak{c} - \pi Z) \equiv \pi^{m-u} G(Z) + \pi^{m-u+1} R(Z) \pmod{\mathfrak{A}_{m+1}},$$

where $f_u(\mathfrak{c} - \pi Z) \equiv \pi^{m-u} G(Z) \pmod{p^{m-u+1}}$ (the congruence being coefficient-wise), where

$$G(Z) = \sum K^{(\omega)}(Z) \Lambda_u^{(\omega)}(Z)$$

where the sum ranges over partitions (ω) of u. And $K^{(\omega)}(Z) = \sum \gamma_{s,(\sigma)} k^{(\sigma)}(Z)$, where the sum ranges over those integers $s \leqq m$ and their partitions (σ) for which $(\sigma^*) = (\omega)$ and for which $\Lambda_s^{(\sigma)}$ is in Γ_0. The $\gamma_{s,(\sigma)} = g_{s,(\sigma)}(\mathfrak{a})$, if $g_{s,(\sigma)}(\mathfrak{a})$ is not in $\mathfrak{p}$ and $\gamma_{s,(\sigma)} = 0$ if $g_{s,(\sigma)}(\mathfrak{a})$ is in $\mathfrak{p}$.

If (σ) and (δ) are either different partitions of the same integer or are partitions of different integers, there is a j such that $\sigma_j \neq \delta_j$. If $(\sigma^*) = (\delta^*)$, then $\mu_1(\sigma_j) \neq \mu_1(\delta_j)$. Hence in $K^{(\omega)}$ the various $\gamma_{s,(\sigma)}$ are coefficients of different monomials. Since $\Lambda_s^{(\sigma)}$ were in Γ_0, the degree of each indeterminate in the $k^{(\sigma)}$, and hence in the $K^{(\omega)}$, is less than q.

Since $f(c - \pi Z)$ is in B_{m+1}, $G(Z) + \pi R(Z)$ must be in B_{u+1}. But $u + 1 \leqq m$, hence $G(Z) + \pi R(Z)$ is in A_{u+1}. Consequently $G(Z)$ is the zero polynomial, and thus the $K^{(\omega)}(Z)$ are the zero polynomial. Hence for those integers s and their partitions (σ) for which $\Lambda_s^{(\sigma)}$ is in Γ_0, $s \leqq m$ and $\sigma^* = u$, we have $g_{s,(\sigma)}(a) \equiv 0 \pmod{p}$. Since this is true for every a in $\mathfrak{B}$, we must have for these s and (σ) that $g_{s,(\sigma)}$ is the zero polynomial. Consequently $f_u(X)$ is the zero polynomial. One can now continue step-wise, and show that for each r, $f_r(X)$ is the zero polynomial. Hence $f(X)$ is in $\mathfrak{A}_{m+1}$.

5. Generalization. If we let $\mathfrak{O}$ be a ring of algebraic integers and let $\mathfrak{m}$ be an ideal in $\mathfrak{O}$ which is not a power of a prime ideal in $\mathfrak{O}$ and consider $B_{\mathfrak{m}}$, or $\mathfrak{B}_{\mathfrak{m}}$, we have considerable more difficulty. If $\mathfrak{m} = \mathfrak{p}_1{}^a \mathfrak{p}_2{}^b \cdots \mathfrak{p}_s{}^c$, $B_{\mathfrak{m}} \neq B_{\mathfrak{p}_1{}^a} B_{\mathfrak{p}_2{}^b} \cdots B_{\mathfrak{p}_s{}^c}$. For let Z be the ring of rational integers and let p and q be primes in Z, then $x(x^{p-1} - 1)(x^{q-1} - 1)$ is in B_{pq}. This illustrates the type of element in $B_{\mathfrak{m}}$. We leave further discussion of $B_{\mathfrak{m}}$ to another time.

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A GENERAL THEORY OF ALGEBRAIC GEOMETRY OVER DEDEKIND DOMAINS, I.*

The Notion of Models.

By Masayoshi Nagata.

In the present sequence of papers, we want to study a general theory of algebraic geometry over a ring, which is a field or a Dedekind domain, under the restriction that the almost finite integral extensions of this ring are finite. (Observe that this condition is satisfied by fields, by complete discrete valuation rings and by Dedekind domain of characteristic zero.)

The writer wishes at first to express his hearty thanks to Professor C. Chevalley, to whose lectures[1] at Kyôto University the writer owes many ideas and who gave the writer many suggestions during the preparation of the present paper.

In Chapter 1, we prove some preliminary results on rings (mainly on spots, which will play an important rôle in our study). In Chapter 2, we study the notion of models of function fields.

In Chapter 1 we first prove the normalization theorem in a generalized form (§ 1) and then we define the notion of spots and study some of their properties; here the notions of affine rings and of function fields are also defined (§§ 2-4). Applying a result in § 4, we prove the finiteness of the derived normal ring of an affine ring in § 5. In § 6, we prove some lemmas on valuation rings.

In Chapter 2, we first introduce the notions of places and of models (§§ 1-2) and then we introduce the notion of specializations (§ 3). Then we study the notion of joins of models (§ 4) and prove the existence of the derived normal model of a model (§ 5). In §§ 6-7, we introduce the Zariski topology on models and in § 8 we introduce the notions of induced model, local model and reduced model. In § 9, we show that under a certain restriction on the function field under consideration, the notion of model is equivalent to the notion of abstract variety in the sense of Weil [12].

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[1] Professor C. Chevalley lectured at Kyôto University in January of 1954. Our definition of models is an adaptation of his to our case. Main results in Chapter 2 of the present paper were shown by him for the case of algebraic geometry over field in his lecture.

Terminology and notations.

Besides the terminology which was used in Nagata [10], we use the following terms: A ring is called *quasi-local* if it has only one maximal ideal. When $\mathfrak{o}$ and $\mathfrak{o}'$ are quasi-local rings, we say that $\mathfrak{o}'$ *dominates* $\mathfrak{o}$ if $\mathfrak{o}$ is a subring of $\mathfrak{o}'$ and if the maximal ideal of $\mathfrak{o}'$ lies over that of $\mathfrak{o}$ (we say that an ideal $\mathfrak{a}'$ of a ring $\mathfrak{o}'$ lies over an ideal $\mathfrak{a}$ of its subring $\mathfrak{o}$ if $\mathfrak{a} = \mathfrak{a}' \cap \mathfrak{o}$). Observe here that domination defines a partial order.

A ring $\mathfrak{o}$ is called a *semi-local* ring if it has only a finite number of maximal ideals and if the topology of $\mathfrak{o}$ introduced by taking all powers of its J-radical as a system of neighborhoods of zero is Hausdorff; a quasi-local semi-local ring is a *local* ring. But since we treat mainly the Noetherian case, we shall mean by local or semi-local ring a *Noetherian* local or semi-local ring, unless the contrary is explicitly stated.

When $\mathfrak{o}$ is an integral domain, the integral closure of $\mathfrak{o}$ in its field of quotients is called the *derived normal ring* of $\mathfrak{o}$. Let $\mathfrak{o}$ be an integral domain. An integral extension $\mathfrak{o}'$ of $\mathfrak{o}$ is said to be *almost finite* if the field of quotients of $\mathfrak{o}'$ is a finite algebraic extension of that of $\mathfrak{o}$ (see [10]); we say that $\mathfrak{o}$ satisfies the *finiteness condition for integral extensions* if every almost finite integral extension of $\mathfrak{o}$ is a finite $\mathfrak{o}$-module. Observe that, if $\mathfrak{o}$ satisfies the finiteness condition for integral extensions, then so does any ring of quotients of $\mathfrak{o}$.

Let $\mathfrak{o}$ be a ring and let S be a set of elements of a ring containing $\mathfrak{o}$. Consider the ring $\mathfrak{o}' = \mathfrak{o}[S]$. Let T be the intersection of the complements of all prime divisors of ideals of $\mathfrak{o}'$ generated by maximal ideals of $\mathfrak{o}$. Then the ring $\mathfrak{o}'_T$ is denoted by $\mathfrak{o}(S)$. Observe that if S is a set of independent elements over a local ring $\mathfrak{o}$ with maximal ideal $\mathfrak{m}$, then $\mathfrak{o}(S) = \mathfrak{o}[S]_{\mathfrak{m}\mathfrak{o}[S]}$. (We shall use mainly this last case.)

Though the notion of rank of rings was defined in [10], we shall repeat it again: A ring $\mathfrak{o}$ is said to be of *rank* n if there exists a chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ of $n+1$ prime ideals $\mathfrak{p}_i$ of $\mathfrak{o}$ and if there exists no such chain with more prime ideals; here the symbol $\subset$ means "included in and different from" (to indicate only "included in," we shall use the symbol $\subseteq$) and prime ideals mean those which are different from the ring. When $\mathfrak{a}$ is an ideal of $\mathfrak{o}$, rank $\mathfrak{o}/\mathfrak{a}$ is called the *co-rank* of $\mathfrak{a}$. The rank of a prime ideal $\mathfrak{p}$ of $\mathfrak{o}$ is defined as the rank of $\mathfrak{o}_{\mathfrak{p}}$; the *rank* of an ideal $\mathfrak{a}$ is the minimum of the ranks of the prime divisors of $\mathfrak{a}$.

Results assumed to be known.

Besides elementary results on fields and rings of polynomials, we need

some results on commutative rings: (1) For the general theory of commutative rings, results which are contained in Nagata [10] are assumed to be known. (2) For the theory of local rings, we assume that the following lemmas are known:

LEMMA 0.1. *If* $\mathfrak{o}$ *is a local ring with maximal ideal* $\mathfrak{m}$, *then the completion* $\mathfrak{o}^*$ *of* $\mathfrak{o}$ *is a local ring with maximal ideal* $\mathfrak{m}\mathfrak{o}^*$. *Further, rank* $\mathfrak{o} =$ *rank* $\mathfrak{o}^*$ *and if* $\mathfrak{a}$ *is an ideal of* $\mathfrak{o}$, *then* $\mathfrak{a}\mathfrak{o}^* \cap \mathfrak{o} = \mathfrak{a}$. (See Krull [5], Cohen [2], Nagata [9], Samuel [11].)

LEMMA 0.2. *Assume that* $\mathfrak{o}$ *is a semi-local ring with maximal ideals* $\mathfrak{p}_1, \cdots, \mathfrak{p}_h$. *Then the completion of* $\mathfrak{o}$ *is the direct sum of the completions of the local rings* $\mathfrak{o}_{\mathfrak{p}_1}, \cdots, \mathfrak{o}_{\mathfrak{p}_h}$. (See Chevalley [1], Nagata [9], Samuel [11].)

(A proof can be given as follows: Set $\mathfrak{m} = \bigcap_i \mathfrak{p}_i$. Then the completion $\mathfrak{o}^*$ of $\mathfrak{o}$ is the limit space of the inverse system $\{\mathfrak{o}/\mathfrak{m}^n; n = 1, 2, \cdots\}$. Since the $\mathfrak{p}_i$'s are maximal ideals, $\mathfrak{o}/\mathfrak{m}^n$ is isomorphic to the direct sum of rings $\mathfrak{o}/\mathfrak{p}_i^n$ $(1 \leqq i \leqq h)$. But $\mathfrak{o}/\mathfrak{p}_i^n = \mathfrak{o}_{\mathfrak{p}_i}/\mathfrak{p}_i^n\mathfrak{o}_{\mathfrak{p}_i}$. Therefore $\mathfrak{o}^*$ is isomorphic to the direct product (sum) of the limit spaces of the inverse systems $\{\mathfrak{o}_{\mathfrak{p}_i}/\mathfrak{p}_i^n\mathfrak{o}_{\mathfrak{p}_i}\}$, whose limits coincide with the completions of the $\mathfrak{o}_{\mathfrak{p}_i}$.)

LEMMA 0.3. *Let* $\mathfrak{a}$ *be an ideal of a semi-local ring* $\mathfrak{o}$ *and let* $\mathfrak{o}^*$ *be the completion of* $\mathfrak{o}$. *Then* $\mathfrak{a}\mathfrak{o}^* \cap \mathfrak{o} = \mathfrak{a}$ and $\mathfrak{o}^*/\mathfrak{a}\mathfrak{o}^*$ *is the completion of* $\mathfrak{o}/\mathfrak{a}$. (See Chevalley [1], Nagata [9], Samuel [11].)

LEMMA 0.4. *Let* $\mathfrak{o}$, $\mathfrak{o}^*$ *and* $\mathfrak{a}$ *be as in Lemma* 0.3. *If* b *is an element of* $\mathfrak{o}$, *then* $\mathfrak{a}\mathfrak{o}^* : b\mathfrak{o}^* = (\mathfrak{a} : b\mathfrak{o})\mathfrak{o}^*$. (See Zariski [15], Nagata [9], Samuel [11].)

COROLLARY. *If an element* a *of* $\mathfrak{o}$ *is not a zero-divisor in* $\mathfrak{o}$, *then* a *is not a zero-divisor in* $\mathfrak{o}^*$. (Chevalley [1])

LEMMA 0.5. *Let* $\mathfrak{o}$ *be a complete local ring with maximal ideal* $\mathfrak{m}$. *Assume that a local ring* $\mathfrak{o}'$ *dominates* $\mathfrak{o}$. *If* $\mathfrak{o}'/\mathfrak{m}\mathfrak{o}'$ *is a finite* $\mathfrak{o}/\mathfrak{m}$-*module, then* $\mathfrak{o}'$ *is a finite* $\mathfrak{o}$-*module and is a complete local ring.* (See Chevalley [1], Cohen [2], Nagata [9], Samuel [11].)

LEMMA 0.6. *Let* $\mathfrak{o}$ *and* $\mathfrak{o}'$ *be semi-local rings which satisfy the following conditions*: 1) $\mathfrak{o}$ *is a subring of* $\mathfrak{o}'$, 2) $\mathfrak{o}'$ *is a finite* $\mathfrak{o}$-*module generated by* $y_1, \cdots, y_n$ *and* 3) *every nonzero element of* $\mathfrak{o}$ *is not a zero-divisor in* $\mathfrak{o}'$. *Then* I) $\mathfrak{o}$ *is a subspace of* $\mathfrak{o}'$, II) *the completion* $\mathfrak{o}'^*$ *of* $\mathfrak{o}'$ *is the module generated by* $y_1, \cdots, y_n$ *over the completion* $\mathfrak{o}^*$ *of* $\mathfrak{o}$, III) *if elements* $x_1, \cdots, x_r$ *of* $\mathfrak{o}'$ *are linearly independent over* $\mathfrak{o}$, *then they are linearly*

independent over $\mathfrak{o}^*$ *and* IV) *if an element* a *of* $\mathfrak{o}^*$ *is not a zero-divisor in* $\mathfrak{o}^*$, *then it is not a zero-divisor in* $\mathfrak{o}'^*$. (See Chevalley [1], Nagata [9], Samuel [11].)

LEMMA 0.7. *A regular local ring is a normal ring.* (See Krull [5], Cohen [2], Nagata [9], Samuel [11].)

LEMMA 0.8. *Let* $\mathfrak{o}$ *be a complete local ring. Assume that* $\mathfrak{o}$ *contains a field or dominates a descrete valuation ring with prime element* p; *in the latter case we asume further that* $p\mathfrak{o}$ *is of rank* 1. *Then there exists an unramified regular local ring* $\mathfrak{r}$ *contained in* $\mathfrak{o}$ *such that* $\mathfrak{o}$ *is a finite* $\mathfrak{r}$*-module.* (See Cohen [2], Samuel [11].) (It was communicated to the writer that a much simplified proof of this lemma is given by Mr. Narita in a forthcoming paper. (Added October, 1955.))

LEMMA 0.9. *If* $\mathfrak{r}$ *is a complete, unramified regular local ring, then any prime ideal of rank* 1 *in* $\mathfrak{r}$ *is principal.* (See Cohen [2], Nagata [9], Samuel [11].)

Remark. This last lemma holds without the assumption that $\mathfrak{r}$ is complete (see Nagata [9]; the proof will be repeated in the second paper of this sequence), as was announced by Mr. Y. Mori in the spring meeting of the Mathematical Society of Japan in 1949.

Numbering.

Numbering of lemmas will begin anew in each section; numbering of propositions and theorems will begin in each chapter. When we refer to a lemma in another section, we shall use notation such as Lemma 1.2.3, the first number, the second one and the third one indicating the number of chapter, section and the lemma respectively; as for the theorems or propositions, we shall use notations like Theorem 1.1, the first number indicating the number of the chapter.

On the restriction of ground rings.

As was stated above, ground rings are assumed to satisfy the finiteness condition for integral extensions. But the definition of spots, affine rings, function fields, models and so on may be given without making use of this condition. When we want to talk about these notions without assuming that the ground ring satisfies the finiteness condition for integral extensions, we shall say: "in the non-restricted case."

Chapter 1. Preliminaries from the Theory of Rings.

Most of the results in the present chapter are not new: Results in § 1 are essentially contained in Nagata [7], [9]. The main results in § 4 is a slight generalization of a result in Nagata [8]. The results in § 6 are well known, but they are basic for the theory of valuation rings.

1. Normalization theorem.

LEMMA 1. *Let $\boldsymbol{k}$ be a field and let $x_1, \cdots, x_n$ be algebraically independent elements over $\boldsymbol{k}$. If y_1 is an element of $\boldsymbol{k}[x_1, \cdots, x_n]$ which is not in $\boldsymbol{k}$, then there exist elements $y_2, \cdots, y_n$ of $\boldsymbol{k}[x_1, \cdots, x_n]$ such that* 1) $y_i = x_i + x_1^{m_i}$ *for some natural number m_i $(i = 2, \cdots, n)$ and* 2) $\boldsymbol{k}[x_1, \cdots, x_n]$ *is integral over $\boldsymbol{k}[y_1, \cdots, y_n]$ (and therefore $y_1, \cdots, y_n$ are algebraically independent over $\boldsymbol{k}$).*

Remark 1. If $\boldsymbol{k}$ contains infinitely many elements, we may replace the first condition by: "$y_2, \cdots, y_n$ are linear combinations of $x_1, \cdots, x_n$ with coefficients in $\boldsymbol{k}$."

Remark 2. For any given natural number r, the m_i's may be selected so as to be multiple of r, as will easily be seen from the proof below.

Proof. We write y_1 as $\sum_i a_i M_i$, where $a_i \varepsilon \boldsymbol{k}$, $a_i \neq 0$ and the M_i's are monomials in $x_1, \cdots, x_n$. We define weights $m_1 = 1$, $m_2, \cdots, m_n$ of x_1, $x_2, \cdots, x_n$ such that one M_i, say M_1, has greater weight than the others. (For example, set $m_i = (d+1)^{i-1}$ for each i, where d is the degree of the polynomial y_1.) Set $y_i = x_i + x_1^{m_i}$ for $i = 2, \cdots, n$. Then y_1 can be written $a_1 x_1^w + f_1 x_1^{w-1} + \cdots + f_w$, where the f_i's are polynomials in $y_2, \cdots, y_n$ with coefficients in $\boldsymbol{k}$ and $w =$ weight M_1. Then these y_i's are the required elements.

PROPOSITION 1. **(Normalization theorem for polynomial rings)** *Let $\boldsymbol{k}$ be a field and let $x_1, \cdots, x_n$ be algebraically independent elements over $\boldsymbol{k}$. If $\mathfrak{a}$ is an ideal of* rank *r in $\boldsymbol{k}[x_1, \cdots, x_n]$, then there exist elements $y_1, \cdots, y_n$ of $\boldsymbol{k}[x_1, \cdots, x_n]$ such that* 1) $\boldsymbol{k}[x_1, \cdots, x_n]$ *is integral over* $\boldsymbol{k}[y_1, \cdots, y_n]$, 2) $\boldsymbol{k}[y_1, \cdots, y_n] \cap \mathfrak{a}$ *is generated by $y_1, \cdots, y_r$ and* 3) $y_{r+j} = x_{r+j} + f_j$ *with f_j in $\pi[x_1, \cdots, x_r]$ for each $j = 1, \cdots, n-r$, where π is the prime integral domain of $\boldsymbol{k}$.*

Remark 1. This condition 3) shows in particular that $\boldsymbol{k}[x_1, \cdots, x_n] = \boldsymbol{k}[x_1, \cdots, x_r, y_{r+1}, \cdots, y_n]$ and that y_{r+j} $(j \geqq 1)$ is in $\pi[x_1, \cdots, x_n]$.

Remark 2. **Assume that $\boldsymbol{k}$ is of characteristic $p \neq 0$. Then taking the natural number r in Remark 2 after Lemma 1 to be equal to p, we see that we can select f_j's to be in $\pi[x_1^p, \cdots, x_r^p]$.**

Proof. When $r=0$, our assertion is evident and we prove our assertion by induction on r. Let $\mathfrak{a}'$ be an ideal contained in $\mathfrak{a}$ and of rank $r-1$. Then there exist elements $y_1, \cdots, y_{r-1}, y'_r, \cdots, y'_n$ of $\boldsymbol{k}[x_1, \cdots, x_n]$ which satisfy the conditions in our assertion for $\mathfrak{a}'$ instead of $\mathfrak{a}$. Since $\mathfrak{a}$ is of rank r, $\mathfrak{a} \cap \boldsymbol{k}[y_1, \cdots, y_{r-1}, y'_r, \cdots, y'_n]$ is of rank r ([10, §8]). Therefore there exists an element y_r of $\mathfrak{a} \cap \boldsymbol{k}[y'_r, \cdots, y'_n]$ which is not zero. Then applying Lemma 1 to y_r and $\boldsymbol{k}[y'_r, \cdots, y'_n]$ we see the existence of $y_{r+1}, \cdots, y_n$ of $\boldsymbol{k}[y'_r, \cdots, y'_n]$ such that i) $y_{r+j} = y'_{r+j} + y'^{m_j}_r$ for some m_j and ii) $\boldsymbol{k}[y'_r, \cdots, y'_n]$ is integral over $\boldsymbol{k}[y_r, \cdots, y_n]$. Then by condition i), we see that condition 3) in our assertion is satisfied by $y_{r+1}, \cdots, y_n$. By condition ii), $\boldsymbol{k}[x_1, \cdots, x_n]$ is integral over $\boldsymbol{k}[y_1, \cdots, y_n]$. Since $\mathfrak{a} \cap \boldsymbol{k}[y_1, \cdots, y_n]$ is of rank r and since $\mathfrak{a}$ contains $y_1, \cdots, y_r$, we see that $\mathfrak{a} \cap \boldsymbol{k}[y_1, \cdots, y_n]$ is generated by $y_1, \cdots, y_r$.

COROLLARY 1. *Let I be an integral domain and let $x_1, \cdots, x_n$ be algebraically independent elements over I. Let $\boldsymbol{k}$ be the field of quotients of I. If $\mathfrak{a}$ is an ideal of $I[x_1, \cdots, x_n]$ such that $\mathfrak{a} \cap I = 0$, then there exist elements $y_1, \cdots, y_n$ of $I[x_1, \cdots, x_n]$ and an element a $(\neq 0)$ of I such that* 1) *$I[a^{-1}, x_1, \cdots, x_n]$ is integral over $I[a^{-1}, y_1, \cdots, y_n]$ and* 2) *$\mathfrak{a}I[a^{-1}, x_1, \cdots, x_n] \cap I[a^{-1}, y_1, \cdots, y_n]$ is generated by $y_1, \cdots, y_r$ with $r = \operatorname{rank} \mathfrak{a}\boldsymbol{k}[x_1, \cdots, x_n]$. Further* 3) *if π is the prime integral domain of I, we can choose $y_{r+1}, \cdots, y_n$ from $\pi[x_1, \cdots, x_n]$.*

Proof. Set $\mathfrak{a}' = \mathfrak{a}\boldsymbol{k}[x_1, \cdots, x_n]$. Take elements $y_1, \cdots, y_n$ as in the above proposition applied to $\mathfrak{a}'$ and $\boldsymbol{k}[x_1, \cdots, x_n]$. Then $y_{r+1}, \cdots, y_n$ are in $\pi[x_1, \cdots, x_n]$ $(\subseteq I[x_1, \cdots, x_n])$. For each $i \leqq r$, there exists an element a_i $(\neq 0)$ of I such that $a_i y_i \varepsilon \mathfrak{a}$, because $y_i \varepsilon \mathfrak{a}'$. Since $\boldsymbol{k}$ is a field, $a_1 y_1, \cdots, a_r y_r$ are as good as $y_1, \cdots, y_r$. Therefore we may assume that $y_1, \cdots, y_r$ are in $\mathfrak{a}$. Since x_i is integral over $\boldsymbol{k}[y_1, \cdots, y_n]$, there exists an element c_i $(\neq 0)$ of I such that $c_i x_i$ is integral over $I[y_1, \cdots, y_n]$ for each i. On the other hand, let $\mathfrak{p}_1, \cdots, \mathfrak{p}_m$ be the set of prime divisors of $\mathfrak{a}$ which contain nonzero elements of I and let b be a nonzero element of I which is contained in all of $\mathfrak{p}_i$. Let a be the product of b and all the c_j's. Then as is easily seen, this a and the above $y_1, \cdots, y_n$ are the required elements.

COROLLARY 2. **(Normalization theorem (for finitely generated rings))**

Assume that a ring $\mathfrak{o}$ *is generated by elements* $x_1, \cdots, x_n$ *over an integral domain* I. *Assume further that no element* a $(\neq 0)$ *of* I *is a zero-divisor in* $\mathfrak{o}$. *Then there exist elements* $z_1, \cdots, z_t$ *of* $\pi[x_1, \cdots, x_n]$ *(where* π *is the prime integral domain) which are algebraically independent over* I *and an element* a $(\neq 0)$ *of* I *such that* $\mathfrak{o}[1/a]$ *is integral over* $I[1/a, z_1, \cdots, z_t]$.

Remark. Observe that if I is a field, then our assumption on I is satisfied and we may take $a = 1$.

Proof. Since $\mathfrak{o}$ is a homomorphic image of a polynomial ring $\mathfrak{o}'$ over I, applying Corollary 1 to $\mathfrak{o}'$ and the kernel of the homomorphism, we prove our assertion.

COROLLARY 3. *Let* $\mathfrak{o}$ *be an integral domain which is finitely generated over a field* $\boldsymbol{k}$. *Then for any prime ideal* $\mathfrak{p}$ *of* $\mathfrak{o}$, rank $\mathfrak{p}$ + co-rank $\mathfrak{p}$ *is equal to the transcendence degree of* $\mathfrak{o}$ *over* $\boldsymbol{k}$ *and* co-rank $\mathfrak{p}$ *is equal to the transcendence degree of* $\mathfrak{o}/\mathfrak{p}$ *over* $\boldsymbol{k}$.

Proof. By Corollary 2 and some results in [10, §§ 4-5], we may assume that $\mathfrak{o}$ (and for the last assertion, $\mathfrak{o}/\mathfrak{p}$) is generated by algebraically independent elements of $\boldsymbol{k}$. Then our proposition shows the validity of Corollary 3.

COROLLARY 4. *Let* $\boldsymbol{k}$ *be a field and let* $x_1, \cdots, x_n$ *be elements of a ring containing* $\boldsymbol{k}$. *Then every maximal ideal* $\mathfrak{m}$ *of* $\boldsymbol{k}[x_1, \cdots, x_n]$ *is generated by* n *elements. Further* $\boldsymbol{k}[x_1, \cdots, x_n]/\mathfrak{m}$ *is algebraic over* $\boldsymbol{k}$. (Zariski [14])

Proof. By Corollary 3, we see that $\boldsymbol{k}[x_1, \cdots, x_n]/\mathfrak{m}$ is algebraic over $\boldsymbol{k}$. Let x'_i be the residue class of x_i modulo $\mathfrak{m}$ for each i and let $f'_i(X_i)$ be the irreducible monic polynomial over $\boldsymbol{k}[x'_1, \cdots, x'_{i-1}]$ which has x'_i as a root. Let f_i be the monic polynomial in x_i with coefficients in $\boldsymbol{k}[x_1, \cdots, x_{i-1}]$ which is obtained from f'_i replacing $x'_1, \cdots, x'_{i-1}, X_i$ by $x_1, \cdots, x_{i-1}, x_i$ respectively. Then we see that $\mathfrak{m}$ is generated by $f_1, \cdots, f_n$.

COROLLARY 5. *Let* I *be a field or a Dedekind domain and let* $x_1, \cdots, x_n$ *be algebraically independent elements over* I. *If* $\mathfrak{p}$ *is a prime ideal of rank* r *in the ring* $\mathfrak{o} = I[x_1, \cdots, x_n]$, *then* $\mathfrak{o}_{\mathfrak{p}}$ *is a regular local ring of rank* r.

Proof. We first assume that I is a field. Let $y_1, \cdots, y_n$ be elements of $\mathfrak{o}$ as in Proposition 1 applied to $\mathfrak{p}$. Then $\mathfrak{o} = I[x_1, \cdots, x_r, y_{r+1}, \cdots, y_n]$ by condition 3) of the proposition. Let K be the field of quotients of $I[y_{r+1}, \cdots, y_n]$. Then $\mathfrak{p}K[x_1, \cdots, x_r]$ is a maximal ideal of $K[x_1, \cdots, x_r]$. Therefore $\mathfrak{p}K[x_1, \cdots, x_r]$ is generated by r elements. Since

$$\mathfrak{o}_{\mathfrak{p}} = K[x_1, \cdots, x_r]_{\mathfrak{p}K[x_1, \cdots, x_r]},$$

this proves our assertion in this case. Now we prove the general case. Set $\mathfrak{q} = \mathfrak{p} \cap I$. If $\mathfrak{q} = 0$, then $\mathfrak{o}_\mathfrak{p}$ contains the field of quotients of I and the assertion follows from the case where I is a field. Therefore we assume that $\mathfrak{q} \neq 0$. Since $I_\mathfrak{q}$ is a principal ideal ring ([10, § 9]) and since $\mathfrak{o}_\mathfrak{p}$ contains $I_\mathfrak{q}$, we may assume that $\mathfrak{q}$ is generated by an element q. Since $I/\mathfrak{q}$ is a field and since $\mathfrak{p}/q\mathfrak{o}$ is of rank $r-1$, $\mathfrak{p}\mathfrak{o}_\mathfrak{p}/q\mathfrak{o}_\mathfrak{p}$ is generated by $r-1$ elements and therefore $\mathfrak{p}\mathfrak{o}_\mathfrak{p}$ is generated by r elements, which proves our assertion.

COROLLARY 6. *Let I be an integral domain and let $x_1, \cdots, x_n$ $(n \geqq 1)$ be algebraically independent elements over I. Then there exists a maximal ideal $\mathfrak{m}$ of $I[x_1, \cdots, x_n]$ such that $\mathfrak{m} \cap I = 0$ if and only if there exists an element a $(\neq 0)$ of I such that $I[1/a]$ is a field.*

Proof. If there exists such an a, then a maximal ideal $\mathfrak{m}$ containing $ax_1 - 1$ meets I only in 0. Conversely, we assume that there exists such a maximal ideal $\mathfrak{m}$. Set $\mathfrak{o} = I[x_1, \cdots, x_n]/\mathfrak{m}$. Then by Corollary 2, there exists an element a $(\neq 0)$ of I such that for a suitable system of algebraically independent elements $y_1, \cdots, y_r$ of $\mathfrak{o}$ over I, $\mathfrak{o}[1/a]$ is integral over $I[1/a, y_1, \cdots, y_r]$. Since $\mathfrak{o}$ is a field, $\mathfrak{o}[1/a] = \mathfrak{o}$. Further, since there is a field which is integral over $I[1/a, y_1, \cdots, y_r]$, $I[1/a, y_1, \cdots, y_r]$ must be a field ([10, § 4]), whence $r = 0$ and $I[1/a]$ is a field (and therefore this is the field of quotients of I).

We have proved at the same time the following

COROLLARY 7. *Assume that $\mathfrak{o}$ is a finitely generated ring over a ring I. If $\mathfrak{m}$ is a maximal ideal of $\mathfrak{o}$, then $\mathfrak{o}/\mathfrak{m}$ is algebraic over $I/(\mathfrak{m} \cap I)$.*

PROPOSITION 2. *Let $x_1, \cdots, x_n$ be algebraically independent elements over a Noetherian integral domain I. If I satisfies the finiteness condition for integral extensions, then so does $I[x_1, \cdots, x_n]$.*

Proof. Let L be a finite extension field of the field of quotients K of $I[x_1, \cdots, x_n]$; we have only to show that every integral extension of $I[x_1, \cdots, x_n]$ contained in L is a finite $I[x_1, \cdots, x_n]$-module. By our assumption on I, we may assume that I is normal. If L is separable over K, then the assertion is obvious ([10, § 5]). When L is inseparable over K, take elements $a_1, \cdots, a_r$ of I and a power q of the characteristic of I such that

$$L' = L(a_1^{1/q}, \cdots, a_r^{1/q}, x_1^{1/q}, \cdots, x_n^{1/q})$$

is separable over

$$K' = K(a_1^{1/q}, \cdots, a_r^{1/q}, x_1^{1/q}, \cdots, x_n^{1/q}).$$

Let I' be the derived normal ring of $I[a_1^{1/q},\cdots,a_r^{1/q}]$. Then I' is finite over I by our assumption, and $I'[x_1^{1/q},\cdots,x_n^{1/q}]$ is finite over $I[x_1,\cdots,x_n]$. Since L' is separable over K' the integral closure of $I'[x_1^{1/q},\cdots,x_n^{1/q}]$ in L' is finite over $I'[x_1^{1/q},\cdots,x_n^{1/q}]$, and is therefore finite over $I[x_1,\cdots,x_n]$. Since $I[x_1,\cdots,x_n]$ is Noetherian, we see that every integral extension of $I[x_1,\cdots,x_n]$ contained in L is finite and we have proved the assertion.

COROLLARY. *If an integral domain* $\mathfrak{o}$ *is finitely generated over a field, then the derived normal ring of* $\mathfrak{o}$ *is a finite* $\mathfrak{o}$*-module.*

2. Definition of spots. An integral domain $\mathfrak{o}$ is called an *affine ring* over a *ground ring* I if I is a field or a Dedekind domain and if $\mathfrak{o}$ is finitely generated over I; here we assume that any ground ring satisfies the finiteness condition for integral extensions (cf. the remark "On the restriction of ground rings" at the beginning of this paper).

Remark. If $\mathfrak{p}$ is a prime ideal of an affine ring $\mathfrak{o}$ over a ground ring I, then $\mathfrak{o}/\mathfrak{p}$ is an affine ring over $I/(\mathfrak{p}\cap I)$; this last ring is I itself or a field and satisfies the finiteness condition for integral extensions.

A field L is called a *function field* over a ground ring I if there exists an affine ring $\mathfrak{o}$ over I such that L is the field of quotients of $\mathfrak{o}$; such $\mathfrak{o}$ is called an affine ring of L.

A ring P is called a *spot* over a ground ring I if there exists an affine ring $\mathfrak{o}$ over I which has a prime ideal $\mathfrak{p}$ such that $P=\mathfrak{o}_{\mathfrak{p}}$; if L is the field of quotients of P, P is called a spot of L.

Remark 1. If P is a spot over a ground ring I and if $\mathfrak{m}$ is the maximal ideal of P, then P is a spot over $I_{(\mathfrak{m}\cap I)}$; this last ring is a field or a discrete valuation ring ([10, § 9]).

Remark 2. If $\mathfrak{p}$ is a prime ideal of a spot P over a ground ring I, then 1) $P_{\mathfrak{p}}$ is a spot over I and 2) $P/\mathfrak{p}$ is a spot over $I/(\mathfrak{p}\cap I)$.

Remark 3. A spot P is a (Noetherian) local integral domain.

A subring B of a spot P is called a *basic ring* of P if 1) P is a spot over B, 2) B is a field or a valuation ring and P dominates B and 3) the residue class field of P is a finite algebraic extension of that of B.

A basic ring which is a field is called a *basic field.*

PROPOSITION 3. *Every spot* P *has a basic ring.*

Proof. Let I be a ground ring of P; by Remark 1, we may assume that

I is a field or a discrete valuation ring dominated by P. Let p be either a prime element of I or zero according to whether I is a valuation ring or a field. Let $\mathfrak{p}$ be the maximal ideal of P and let $x_1, \cdots, x_d$ be elements of P whose residue classes modulo $\mathfrak{p}$ form a transcendence base of $P/\mathfrak{p}$ over I/pI. Since I is a field or a valuation ring, $x_1, \cdots, x_d$ are algebraically independent over I, $pI[x_1, \cdots, x_d]$ is a prime ideal and $\mathfrak{p} \cap I[x_1, \cdots, x_d] = pI[x_1, \cdots, x_d]$. Set $B = I(x_1, \cdots, x_d)$. Then B is a field or a valuation ring which is dominated by P and the residue class field of P is a finite algebraic extension of that of B. Further B satisfies the finiteness condition for integral extensions by virtue of Proposition 2. Therefore B is a basic ring of P.

A spot P is said to be of *the first kind* if it has a basic field; otherwise, P is said to be of *the second kind.*

Remark. A spot of the first kind may have a basic ring which is not a field.

3. Dimension and rank of spots. Let L be a function field over a ground ring I. The *dimension* of L over I (in symbols, $\dim_I L$ or merely $\dim L$) is n or $n+1$ according to whether I is a field or not, where n is the transcendence degree of L over I. The *dimension* of a spot P over a ground ring I (in symbols, $\dim_I P$ or merely $\dim P$) is defined to be the dimension of the residue class field of P modulo its maximal ideal $\mathfrak{m}$ over $I/(\mathfrak{m} \cap I)$.

LEMMA 1. *Let $x_1, \cdots, x_n$ be algebraically independent elements over a Dedekind domain I and let $\mathfrak{m}$ be a maximal ideal of the ring $\mathfrak{o} = I[x_1, \cdots, x_n]$. Assume that $\mathfrak{m} \cap I \neq 0$. Then $\mathfrak{o}_{\mathfrak{m}}$ is a regular local ring of rank $n+1$; if $\mathfrak{q}$ is a prime ideal contained in $\mathfrak{m}$, then* rank $\mathfrak{q}$ + co-rank $\mathfrak{q} = n+1$ *and* co-rank $\mathfrak{q}$ = co-rank $\mathfrak{q}\mathfrak{o}_{\mathfrak{m}}$.

Proof. The first assertion follows from Corollary 5 to Proposition 1. Since $\mathfrak{o}_{\mathfrak{m}}$ is a regular local ring of rank $n+1$, rank $\mathfrak{q}\mathfrak{o}_{\mathfrak{m}}$ + co-rank $\mathfrak{q}\mathfrak{o}_{\mathfrak{m}} = n+1$ (by a result due to Krull [5]; a proof will be given in the appendix at the end of the present paper). By the definition of rank, rank $\mathfrak{q}$ = rank $\mathfrak{q}\mathfrak{o}_{\mathfrak{m}}$. Since every maximal ideal of $\mathfrak{o}$ is at most of rank $n+1$, co-rank $\mathfrak{q} \leqq n+1$ $-$ rank $\mathfrak{q}$, while, by the definition of co-rank, co-rank $\mathfrak{q} \geqq$ co-rank $\mathfrak{q}\mathfrak{o}_{\mathfrak{m}}$. Therefore co-rank $\mathfrak{q}$ = co-rank $\mathfrak{q}\mathfrak{o}_{\mathfrak{m}}$ and rank $\mathfrak{q}$ + co-rank $\mathfrak{q} = n+1$.

THEOREM 1. *Let $\mathfrak{o}$ be an affine ring over a ground ring I. Let $0 = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_r$ be a maximal chain of prime ideals $\mathfrak{p}_i$ (that is, each $\mathfrak{p}_i/\mathfrak{p}_{i-1}$ is of rank 1 and $\mathfrak{p}_r$ is maximal). If $\mathfrak{p}_r \cap I = 0$, then r is equal to the*

transcendence degree of $\mathfrak{o}$ *over* I; *if* $\mathfrak{p}_r \cap I \neq 0$, *then* $r-1$ *is equal to the transcendence degree of* $\mathfrak{o}$ *over* I.

Proof. When $\mathfrak{p}_r \cap I = 0$, we may assume that I is a field. Then our assertion follows easily from Corollary 3 to Proposition 1. Therefore we assume that $\mathfrak{p}_r \cap I \neq 0$. Let $x_1, \cdots, x_n$ be algebraically independent elements over I such that there exists a homomorphism ϕ from $I[x_1, \cdots, x_n]$ onto $\mathfrak{o}$; let $\mathfrak{q}$ be the kernel of ϕ. Then we have rank $\mathfrak{q}$ + rank $\mathfrak{o} = n+1$, by Lemma 1. Let $\boldsymbol{k}$ be the field of quotients of I. Since $\mathfrak{q} \cap I = 0$, $I[x_1, \cdots, x_n]_{\mathfrak{q}}$ contains $\boldsymbol{k}$. Therefore we have $n -$ rank $\mathfrak{q}$ = transcendence degree of $\mathfrak{o}$ over I; we denote this number by t. Since rank $\mathfrak{q}$ + rank $\mathfrak{o} = n+1$, we have rank $\mathfrak{o} = t+1$. Therefore $r \leqq t+1$. Therefore, when $t=0$, our assertion is evident. Thus we will prove our assertion by induction on t. Assume that $t \geqq 1$. By Lemma 1, rank $\mathfrak{o}$ = rank $\mathfrak{o}_{\mathfrak{p}_r}$. Therefore $r > 1$.

1) When $\mathfrak{p}_{r-1} \cap I \neq 0$: $\mathfrak{o}/\mathfrak{p}_{r-1}$ is an affine ring over the field $I/(\mathfrak{p}_{r-1} \cap I)$, which shows that $\mathfrak{o}/\mathfrak{p}_{r-1}$ has transcendence degree 1 over $I/(\mathfrak{p}_{r-1} \cap I)$. Therefore there exist a basic ring B (defined even in the non-restricted case) of $\mathfrak{o}_{\mathfrak{p}_{r-1}}$ which is of transcendence degree 1 over I (as in the proof of Proposition 3). Set $\mathfrak{o}' = B[\mathfrak{o}]$. Then $0 = \mathfrak{p}_0\mathfrak{o}' \subset \mathfrak{p}_1\mathfrak{o}' \subset \cdots \subset \mathfrak{p}_{r-1}\mathfrak{o}'$ is a maximal chain of prime ideals of $\mathfrak{o}'$, because $\mathfrak{o}'$ is a ring of quotients of $\mathfrak{o}$ (see the construction of B in the proof of Proposition 3); that $\mathfrak{p}_{r-1}\mathfrak{o}'$ is maximal follows from the fact that $\mathfrak{o}/\mathfrak{p}_{r-1}$ is an affine ring over the field $I/(\mathfrak{p}_{r-1} \cap I)$. Since $\mathfrak{o}'$ has transcendence degree $t-1$ over B, we have $r-1=t$ by our induction assumption. Thus, this case is settled.

2) When $\mathfrak{p}_{r-1} \cap I = 0$: Since $r > 1$, $\mathfrak{o}/\mathfrak{p}_{r-1}$ has a transcendence degree less than that of $\mathfrak{o}$ over I. Therefore by our induction assumption, $\mathfrak{o}/\mathfrak{p}_{r-1}$ is algebraic over I. Set $\mathfrak{o}' = \boldsymbol{k}[\mathfrak{o}]$. Then $0 = \mathfrak{p}_0\mathfrak{o}' \subset \mathfrak{p}_1\mathfrak{o}' \subset \cdots \subset \mathfrak{p}_{r-1}\mathfrak{o}'$ is a maximal chain of prime ideal in $\mathfrak{o}'$. Therefore $r-1=t$ by Corollary 3 to Proposition 1. Thus we have settled this case, too.

Corollary 1. *Let P be a spot of a function field L. If I is a basic ring of P, then* $\dim_I L = \operatorname{rank} P$.

Corollary 2. *If $\mathfrak{p}$ is a prime ideal of a spot P, then* rank $\mathfrak{p}$ + co-rank $\mathfrak{p}$ = rank P.

Corollary 3. *Let $\mathfrak{o}$ be an affine ring of a function field over a ground ring I. If $\mathfrak{p}$ is a prime ideal of $\mathfrak{o}$, then* rank $\mathfrak{p} + \dim_I \mathfrak{o}_{\mathfrak{p}} = \dim_I L$.

4. Analytical unramifiedness of spots. A semi-local integral domain is said to be *analytically unramified* if its completion has no niloptent elements.

A prime ideal $\mathfrak{p}$ of a semi-local ring $\mathfrak{o}$ is said to be *analytically unramified* if $\mathfrak{o}/\mathfrak{p}$ is analytically unramified.

LEMMA 1. *Let $\mathfrak{o}$ be a normal semi-local ring. Assume that a prime ideal $\mathfrak{p}$ of rank 1 in $\mathfrak{o}$ is analytically unramified. Let $\mathfrak{o}^*$ be the completion of $\mathfrak{o}$. Then for every prime divisor $\mathfrak{p}^*$ of $\mathfrak{p}\mathfrak{o}^*$, $\mathfrak{o}^*_{\mathfrak{p}^*}$ is a valuation ring.* (Zariski [15])

Remark. This result does not show that $\mathfrak{o}^*$ is an integral domain, but shows that $\mathfrak{p}^*$ contains only one prime divisor $\mathfrak{P}^*$ of zero in $\mathfrak{o}^*$ and that the primary component of zero belonging to $\mathfrak{P}^*$ coincides with $\mathfrak{P}^*$ (and the ring of quotients of $\mathfrak{o}^*/\mathfrak{P}^*$ with respect to $\mathfrak{p}^*/\mathfrak{P}^*$ is a valuation ring).

Proof. Let w be an element of $\mathfrak{p}$ which is not in $\mathfrak{p}^2\mathfrak{o}_{\mathfrak{p}}$. Since $\mathfrak{o}$ is a normal ring, $\mathfrak{o}_{\mathfrak{p}}$ is a discrete valuation ring ([10, § 9]) and $w\mathfrak{o}:\mathfrak{p}$ is not contained in $\mathfrak{p}$. Let b be an element of $w\mathfrak{o}:\mathfrak{p}$ which is not in $\mathfrak{p}$ and let a^* be an element of $\mathfrak{o}^*$ which is not in $\mathfrak{p}^*$ but is in every other prime divisor of $\mathfrak{p}\mathfrak{o}^*$ (such a^* exists because $\mathfrak{p}$ is analytically unramified). Set $c^* = a^*b$. Then c^* is not in $\mathfrak{p}^*$ and $\mathfrak{p}^*c^* \subseteq \mathfrak{o}^*\mathfrak{p}b \subseteq w\mathfrak{o}^*$. Therefore we see that $\mathfrak{p}^*\mathfrak{o}^*_{\mathfrak{p}^*} \subseteq w\mathfrak{o}^*_{\mathfrak{p}^*}$, whence $\mathfrak{p}^*\mathfrak{o}^*_{\mathfrak{p}^*} = w\mathfrak{o}^*_{\mathfrak{p}^*}$. Since w is not a zero-divisor in $\mathfrak{o}^*$ (Corollary to Lemma 0.4), $\mathfrak{p}^*$ properly contains a prime divisor $\mathfrak{P}^*$ of zero. Since in any Noetherian ring, a principal ideal generated by an element of its J-radical cannot properly contain any prime ideal other than zero ([10, § 6]), we see that $\mathfrak{P}^*$ must be the kernel of the natural homomorphism from $\mathfrak{o}^*$ into $\mathfrak{o}^*_{\mathfrak{p}^*}$. Now, since $\mathfrak{p}^*\mathfrak{o}^*_{\mathfrak{p}^*}$ is principal and since $\mathfrak{o}^*_{\mathfrak{p}^*}$ is a local integral domain which is not a field, $\mathfrak{o}^*_{\mathfrak{p}^*}$ must be a valuation ring.

LEMMA 2. *Let $\mathfrak{o}$ be a normal semi-local ring and let $\mathfrak{o}^*$ be its completion. Assume that for an element t of $\mathfrak{o}$, which is neither zero nor unit in $\mathfrak{o}$, every prime divisor of $t\mathfrak{o}$ is analytically unramified. If an element u is integral over $\mathfrak{o}^*$ and if tu is in $\mathfrak{o}^*$, then u is also in $\mathfrak{o}^*$.* (Zariski [16])

Proof. Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_r$ be all the prime divisors of $t\mathfrak{o}$; they are of rank 1 ([10, § 9]). Let $\mathfrak{p}^*_{ij}$ $(j = 1, \cdots, n(i))$ be all the prime divisors of $\mathfrak{p}_i\mathfrak{o}^*$. We set $S = \{a; a \in \mathfrak{o}, a \notin \mathfrak{p}_i \text{ for every } i\}$.[2] Then since $\mathfrak{o}$ is a normal ring, $\mathfrak{o}_S$ is a semi-local Dedekind domain and is a principal ideal ring ([10, § 9]). Let x_i be an element of $\mathfrak{o}$ such that $\mathfrak{p}_i\mathfrak{o}_S = x_i\mathfrak{o}_S$. Let e_i be natural numbers such that $t\mathfrak{o}_S = x_1^{e_1} \cdots x_r^{e_r}\mathfrak{o}_S$. Now we have only to show that for some element s of S, tus is divisible by $x_1^{e_1} \cdots x_r^{e_r}$ in $\mathfrak{o}^*$. For, once this is done, the proof concludes as follows: since $t\mathfrak{o}:s\mathfrak{o} = t\mathfrak{o}$, we see that $t\mathfrak{o}^*:s\mathfrak{o}^* = t\mathfrak{o}^*$ (Lemma 0.4). Since $x_1^{e_1} \cdots x_r^{e_r}\mathfrak{o} \subseteq t\mathfrak{o}$, we see that $x_1^{e_1} \cdots x_r^{e_r}\mathfrak{o}^* \subseteq t\mathfrak{o}^*$.

[2] The notation $\{a; P\}$ denotes the set of a which satisfy P.

Therefore the fact that $tus \in x_1^{e_1} \cdots x_r^{e_r}\mathfrak{o}^*$ shows that $tus \in t\mathfrak{o}^*$ and tu is in $t\mathfrak{o}^* : s\mathfrak{o}^* = t\mathfrak{o}^*$. Then since t is not a zero-divisor in $\mathfrak{o}^*$ (Corollary to Lemma 0.4), u is in $\mathfrak{o}^*$. Now we proceed to show that tus is in $x_1^{e_1} \cdots x_r^{e_r}\mathfrak{o}^*$ for some s in S. Let w_{ij} be the valuation of the field of quotients of $\mathfrak{o}^*_{\mathfrak{p}^*_{ij}}$ with $\mathfrak{o}^*_{\mathfrak{p}^*_{ij}}$ as valuation ring and $w_{ij}(x_i) = 1$. Let ϕ_{ij} be the natural homomorphism from $\mathfrak{o}^*$ into $\mathfrak{o}^*_{\mathfrak{p}^*_{ij}}$. Let $f_1, \cdots, f_r$ be non-negative integers satisfying the following condition: tus is in $x_1^{f_1} \cdots x_r^{f_r}\mathfrak{o}^*$ for some s of S but for any s of S and for any i, tus is not in $x_1^{f_1} \cdots x_{i-1}^{f_{i-1}} x_i^{f_i+1} x_{i+1}^{f_{i+1}} x_r^{f_r}\mathfrak{o}^*$. Then it is sufficient to show that $f_i \geqq e_i$. Assume the contrary, for instance that $f_1 < e_1$. We take an element s of S such that tus is in $x_1^{f_1} \cdots x_r^{f_r}\mathfrak{o}^*$ and take an element z of $\mathfrak{o}^*$ such that $tus = x_1^{f_1} \cdots x_r^{f_r} z$. Since the kernel of ϕ_{ij} is a prime ideal and since $\mathfrak{o}^*_{\mathfrak{p}^*_{ij}}$ is a normal ring, we may regard ϕ_{ij} as a homomorphism of the total quotient ring $\mathfrak{o}^{*\prime}$ of $\mathfrak{o}^*$ into the field of quotients of $\mathfrak{o}^*_{\mathfrak{p}^*_{ij}}$. Then since u is integral over $\mathfrak{o}^*$, $\phi_{ij}(u)$ is in $\mathfrak{o}^*_{\mathfrak{p}^*_{ij}}$, whence $w_{ij}(\phi_{ij}(u)) \geqq 0$. Since $w_{1j}(\phi_{1j}(t)) = e_1 > f_1 = w_{1j}(\phi_{1j}(x_1^{f_1} \cdots x_r^{f_r}))$, we have $w_{1j}(\phi_{1j}(z)) \geqq 1$. This shows that $\phi_{1j}(z)$ is in $\phi_{1j}(\mathfrak{p}^*_{1j})$ and therefore z is in $\mathfrak{p}^*_{1j}$ for every j because z is in $\mathfrak{o}^*$ (observe that the kernel of ϕ_{1j} is contained in $\mathfrak{p}^*_{1j}$). Since $x_1\mathfrak{o}^*_S = \bigcap_j \mathfrak{p}^*_{1j}\mathfrak{o}^*_S$, z is in $x_1\mathfrak{o}^*_S$ and therefore there exists an element s' of S such that $zs' = x_1 z'$ with an element z' of $\mathfrak{o}^*$. Thus we see that $tus'' = x_1^{f_1+1} x_2^{f_2} \cdots x_r^{f_r} z'$ with $s'' = ss'$ (which is in S). This is a contradiction and we have $e_i \leqq f_i$ for every i. Thus the lemma is proved.

LEMMA 3. *Let $\mathfrak{r}$ be a normal local ring whose completion is a normal ring. Let L be a finite separable extension of the field of quotients R of $\mathfrak{r}$ and let $\mathfrak{J}$ be the integral closure of $\mathfrak{r}$ in L. Assume that every prime ideal $\mathfrak{p}$ of rank 1 in $\mathfrak{J}$ is analytically unramified. Then the completion of $\mathfrak{J}$ is integrally closed, that is, for every maximal ideal $\mathfrak{m}$ of $\mathfrak{J}$, the completion of $\mathfrak{J}_\mathfrak{m}$ is a normal ring.* (Zariski [16])

Proof. Let a be an element of $\mathfrak{J}$ such that $L = R(a)$. Let d be the discriminant of the irreducible monic polynomial over $\mathfrak{r}$ which has a as a root. On the other hand, let $\mathfrak{r}^*$ and $\mathfrak{J}^*$ be the completions of $\mathfrak{r}$ and $\mathfrak{J}$ respectively and denote by $\mathfrak{J}^{*\prime}$ the integral closure of $\mathfrak{J}^*$ in its total quotient ring. Since $\mathfrak{J}$ is a finite $\mathfrak{r}$-module, the integral closure of $\mathfrak{r}^*[a]$ in its total quotient ring coincides with $\mathfrak{J}^{*\prime}$ (Lemma 0.6). Therefore we see that $d\mathfrak{J}^{*\prime} \subseteq \mathfrak{r}^*[a]$ ([10, § 5]). By Lemma 2, we see that $\mathfrak{J}^{*\prime} = \mathfrak{J}^*$ and therefore $\mathfrak{J}^*$ is integrally closed. Since $\mathfrak{J}^*$ is integrally closed and is Noetherian, we see that $\mathfrak{J}^*$ is the direct sum of normal rings. On the other hand, since $\mathfrak{J}^*$ is the direct sum of completions of $\mathfrak{J}_\mathfrak{m}$, where $\mathfrak{m}$ runs over all maximal ideals of $\mathfrak{J}$ (Lemma 0.2), we see that the completion of $\mathfrak{J}_\mathfrak{m}$ is a normal ring for every maximal ideal $\mathfrak{m}$ of $\mathfrak{J}$.

LEMMA 4. *Let $\mathfrak{o}$ be a normal local ring and assume that its completion $\mathfrak{o}^*$ is an integral domain. Let L be the field of quotients of $\mathfrak{o}$. Assume that a local ring $\mathfrak{o}'$ which is a subring of L satisfies following conditions*: 1) *$\mathfrak{o}'$ dominates $\mathfrak{o}$*, 2) *$\mathfrak{o}'/\mathfrak{m}'$ is a finite algebraic extension of $\mathfrak{o}/\mathfrak{m}$, where $\mathfrak{m}$ and $\mathfrak{m}'$ denote the maximal ideals of $\mathfrak{o}$ and $\mathfrak{o}'$ respectively*, 3) *$\mathfrak{m}\mathfrak{o}'$ is a primary ideal belonging to $\mathfrak{m}'$ and* 4) $\operatorname{rank}\mathfrak{o}' = \operatorname{rank}\mathfrak{o}$. *Then $\mathfrak{o}$ coincides with $\mathfrak{o}'$.*

Proof. Let $\mathfrak{o}'^*$ be the completion of $\mathfrak{o}'$. Since $\mathfrak{m}' \cap \mathfrak{o} = \mathfrak{m}$, we see that $\mathfrak{m}^i \subseteq \mathfrak{o} \cap \mathfrak{m}'^i$ for every i. Therefore there exists a natural homomorphism ϕ from $\mathfrak{o}^*$ into $\mathfrak{o}'^*$. Set $\mathfrak{o}^{**} = \phi(\mathfrak{o}^*)$, (this is the closure of $\mathfrak{o}$ in $\mathfrak{o}'^*$ in the topology of $\mathfrak{o}'^*$). By conditions 2) and 3), we see that $\mathfrak{o}'^*$ is a finite $\mathfrak{o}^{**}$-module (Lemma 0.5). Therefore $\operatorname{rank}\mathfrak{o}^{**} = \operatorname{rank}\mathfrak{o}'^*$ ([10, § 8]). Since $\operatorname{rank}\mathfrak{o} = \operatorname{rank}\mathfrak{o}'$, $\operatorname{rank}\mathfrak{o}^* = \operatorname{rank}\mathfrak{o}'^*$ (Lemma 0.1) and $\operatorname{rank}\mathfrak{o}^* = \operatorname{rank}\mathfrak{o}^{**}$. Since $\mathfrak{o}^*$ is an integral domain, we see that ϕ is an isomorphism. Thus we see that $\mathfrak{o}$ is a subspace of $\mathfrak{o}'$ and $\mathfrak{o}'^*$ is integral over $\mathfrak{o}^*$. Now let a/b $(a, b \in \mathfrak{o})$ be an element of $\mathfrak{o}'$. Since a/b is integral over $\mathfrak{o}^*$, there exist elements $c^*_1, \cdots, c^*_n$ of $\mathfrak{o}^*$ such that $(a/b)^n + c^*_1(a/b)^{n-1} + \cdots + c^*_n = 0$ and therefore $a^n + ba^{n-1}c^*_1 + \cdots + b^nc^*_n = 0$, which shows that a^n is in the ideal generated by $ba^{n-1}, \cdots, b^n$. Since $(\sum_1^n b^i a^{n-i}\mathfrak{o}^*) \cap \mathfrak{o} = \sum_1^n b^i a^{n-i}\mathfrak{o}$ (Lemma 0.1), we see that there exist elements $c_1, \cdots, c_n$ of $\mathfrak{o}$ such that $a^n + ba^{n-1}c_1 + \cdots + b^nc_n = 0$. Therefore $(a/b)^n + c_1(a/b)^{n-1} + \cdots + c_n = 0$, that is, a/b is integral over $\mathfrak{o}$. Since $\mathfrak{o}$ is normal, a/b is in $\mathfrak{o}$, which proves Lemma 4.

Now we come to the important

THEOREM 2. *Any spot is analytically unramified. Further the completion of a normal spot is a normal ring.*

Proof. Let P be a spot of rank r. We prove our assertion by induction on r. When $r = 0$, our assertion is obvious. Assume that $r > 0$ and that the theorem is true for spots of rank $\leqq r - 1$. We first remark the following fact. Let $\mathfrak{o}$ be a semi-local integral domain. If for every maximal ideal $\mathfrak{m}$ of $\mathfrak{o}$, $\mathfrak{o}_{\mathfrak{m}}$ is analytically unramified, then $\mathfrak{o}$ is also analytically unramified (the proof follows easily from the fact that the completion of $\mathfrak{o}$ is the direct sum of completions of rings $\mathfrak{o}_{\mathfrak{m}}$). Therefore our induction assumption means that if $\mathfrak{o}$ is a semi-local integral domain of rank r such that for any maximal ideal $\mathfrak{m}$ of $\mathfrak{o}$, $\mathfrak{o}_{\mathfrak{m}}$ is a spot and if $\mathfrak{p}$ is a nonzero prime ideal of $\mathfrak{o}$, then $\mathfrak{p}$ is analytically unramified. Let B be a basic ring of P and let $x_1, \cdots, x_r$ be a system of parameters of P, where if B is not a field, we choose x_1 to be a prime element of B. By Corollary 1 to Theorem 1, the field of quotients L

of P is algebraic over $B[x_1,\cdots,x_r]$. Let $\mathfrak{J}$ be the integral closure of $B[x_1,\cdots,x_r]$ in L. Further set $P'=P[\mathfrak{J}]$. $\mathfrak{J}$ is a finite $B[x_1,\cdots,x_r]$-module by Proposition 2 and P' is a finite P-module.

(a) When L is separable over $B[x_1,\cdots,x_r]$, let $\mathfrak{m}'$ be an arbitrary maximal ideal of P' and set $\mathfrak{m}=\mathfrak{m}'\cap\mathfrak{J}$. Then by Lemma 3 and by our induction assumption, we see that the completion of $\mathfrak{J}_{\mathfrak{m}}$ is a normal ring. Since $x_1,\cdots,x_r$ is a system of parameters of $P'_{\mathfrak{m}'}$ and of $\mathfrak{J}_{\mathfrak{m}}$, we see that $P'_{\mathfrak{m}'}=\mathfrak{J}_{\mathfrak{m}}$ by Lemma 4 and Theorem 1. Since the completion of P' is the direct sum of the completions of the rings $P'_{\mathfrak{m}'}$, P' is analytically unramified. Since P' is a finite P-module, P is a subspace of P' (Lemma 0.6) and P is also analytically unramified. When P is normal, P' coincides with P and its completion is a normal ring.

(b) Next we assume that L is not separable over $B[x_1,\cdots,x_r]$. Take elements $a_1,\cdots,a_s$ of B and a power q of the characteristic of B such that

$$L'=L(a_1^{1/q},\cdots,a_s^{1/q},x_1^{1/q},\cdots,x_r^{1/q})$$

is separable over

$$B[a_1^{1/q},\cdots,a_s^{1/q},x_1^{1/q},\cdots,x_r^{1/q}].$$

Let B' be the derived normal ring of $B[a_1^{1/q},\cdots,a_s^{1/q}]$ and let $\mathfrak{J}'$ be the integral closure of $B[x_1,\cdots,x_r]$ in L'. Since L' is separable over $B'[x^{1/q},\cdots,x_r^{1/q}]$, for every maximal ideal $\mathfrak{m}''$ of $P''=P[\mathfrak{J}']$ the completion of $P''_{\mathfrak{m}''}$ is a normal ring and $P''_{\mathfrak{m}''}=\mathfrak{J}'_{(\mathfrak{m}''\cap\mathfrak{J}')}$ (by our observation in (a) above). By Proposition 2, $\mathfrak{J}'$ is a finite $B[x_1,\cdots,x_r]$-module and P'' is a finite P-module, which shows that P is a subspace of P''. Therefore we see that P is also analytically unramified. Assume that P is normal. Let $b_1,\cdots,b_t$ be elements of $\mathfrak{J}$ which are maximally linearly independent over $B[x_1,\cdots,x_r]$ and such that the module generated by $b_1,\cdots,b_t$ over $B[x_1,\cdots,x_r]$ is a ring; let $c_1,\cdots,c_u$ be elements of $\mathfrak{J}'$ which are maximally linearly independent over $B[x_1,\cdots,x_r,b_1,\cdots,b_t]$ and such that the module generated by $c_1,\cdots,c_u$ over $B[x_1,\cdots,x_r,b_1,\cdots,b_t]$ is a ring. Since $\mathfrak{J}'$ is a finite $B[x_1,\cdots,x_r]$-module, there exists an element d $(\neq 0)$ of $B[x_1,\cdots,x_r]$ such that $d\mathfrak{J}'\subseteq B[x_1,\cdots,x_r,b_1,\cdots,b_t,c_1,\cdots,c_u]$. Let P^*, P''^* and $\mathfrak{r}^*$ be the completions of P, P'' and $B[x_1,\cdots,x_r]_{(x_1,\cdots,x_r)}$ respectively. Then we have $dP''^*\subseteq\mathfrak{r}^*[b_1,\cdots,b_t,c_1,\cdots,c_u]$. Let $\mathfrak{S}$ be the integral closure of P^* in its total quotient ring. Since P''^* is integrally closed, $\mathfrak{S}$ is contained in P''^*. Therefore $d\mathfrak{S}\subseteq\mathfrak{r}^*[b_1,\cdots,b_t,c_1,\cdots,c_u]$. Since $c_1,\cdots,c_u$ are linearly independent over $\mathfrak{r}^*[b_1,\cdots,b_t]$ (Lemma 0.6), we see that $d\mathfrak{S}\subseteq\mathfrak{r}^*[b_1,\cdots,b_t]$ and $d\mathfrak{S}\subseteq P^*$. Therefore by Lemma 2 and by our induction assumption, we have $P^*=\mathfrak{S}$ and P^* is a normal ring. Thus the proof is completed.

COROLLARY. *If P is a spot, then the derived normal ring of P is a finite P-module.*

Proof. The ring P' in the above proof is a normal ring and is a finite module over P. Therefore P' is the derived normal ring of P, which proves our assertion.

Remark. It is known that if a semi-local integral domain $\mathfrak{o}$ is analytically unramified, then the derived normal ring of $\mathfrak{o}$ is a finite $\mathfrak{o}$-module. (Nagata [8]; a proof will also be given in the second paper of the present sequence.)

5. Finiteness of derived normal rings of affine rings.

THEOREM 3. *If $\mathfrak{o}$ is an affine ring, then the derived normal ring $\mathfrak{o}'$ of $\mathfrak{o}$ is a finite $\mathfrak{o}$-module.*

Proof. Let I be a groud ring of $\mathfrak{o}$. Then by Corollary 2 to Proposition 1, there exist an element a $(\neq 0)$ of I and elements $y_1, \cdots, y_n$ of $\mathfrak{o}$ which are algebraically independent over I such that $\mathfrak{o}[1/a]$ is integral over $I[1/a, y_1, \cdots, y_n]$. Let $\mathfrak{o}''$ be the integral closure of $I[1/a, y_1, \cdots, y_n]$ in the field of quotients L of $\mathfrak{o}$. Then by Proposition 2, $\mathfrak{o}''$ is a finite $I[1/a, y_1, \cdots, y_n]$-module. If a is a unit in $\mathfrak{o}$, then $\mathfrak{o}'' = \mathfrak{o}'$ and $\mathfrak{o}'$ is a finite $\mathfrak{o}$-module. Therefore we treat the case where a is non-unit in $\mathfrak{o}$. Since $\mathfrak{o}''$ is a finite $I[1/a, y_1, \cdots, y_n]$-module, there exist a finite number of elements $c_1, \cdots, c_r$ in $\mathfrak{o}'$ such that $\mathfrak{o}'' = \mathfrak{o}[1/a, c_1, \cdots, c_r]$ (because $\mathfrak{o}'' = \mathfrak{o}'[1/a]$). Set $\mathfrak{o}_1 = \mathfrak{o}[c_1, \cdots, c_r]$ (which is a finite $\mathfrak{o}$-module). Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_s$ be all the minimal prime divisors of $a\mathfrak{o}_1$. Then there exist a finite number of elements $c'_1, \cdots, c'_t$ in $\mathfrak{o}'$ such that $(\mathfrak{o}_1)_{\mathfrak{p}_i}[c'_1, \cdots, c'_t]$ is a normal ring for every i, by the corollary to Theorem 2. Set $\mathfrak{o}_2 = \mathfrak{o}_1[c'_1, \cdots, c'_t]$. Now we prove the following two lemmas:

LEMMA 1. *For any ring $\mathfrak{s}$ such that $\mathfrak{o}_2 \subseteq \mathfrak{s} \subseteq \mathfrak{o}'$ and for any prime ideal $\mathfrak{p}$ of rank 1 in $\mathfrak{s}$, the ring $\mathfrak{s}_{\mathfrak{p}}$ is a normal ring.*

Proof. If $a \notin \mathfrak{p}$, then $\mathfrak{s}_{\mathfrak{p}}$ contains $1/a$ and therefore $\mathfrak{s}_{\mathfrak{p}}$ is a ring of quotients of $\mathfrak{o}''$, which shows that $\mathfrak{s}_{\mathfrak{p}}$ is a normal ring in this case. When $\mathfrak{p}$ contains a, we set $\mathfrak{p}' = \mathfrak{p} \cap \mathfrak{o}_2$. Then $\mathfrak{p}'$ is of rank 1 by Theorem 1. Therefore $(\mathfrak{o}_2)_{\mathfrak{p}'}$ is a normal ring and it contains $\mathfrak{o}'$. Therefore $\mathfrak{s}_{\mathfrak{p}}$ is a ring of quotients of $\mathfrak{o}'$ and is a normal ring.

LEMMA 2. *With the same $\mathfrak{s}$ as in Lemma 1, let $\mathfrak{q}_1, \cdots, \mathfrak{q}_u$ be all the imbedded prime divisors of $a\mathfrak{s}$. If $d_1, \cdots, d_v$ are elements of $\mathfrak{o}'$ such that*

$\mathfrak{S}_{\mathfrak{q}_i}[d_1, \cdots, d_v]$ *is a normal ring for every i and if* $\mathfrak{q}'$ *is an imbedded prime divisor of* $a\mathfrak{S}'$, *where* $\mathfrak{S}' = \mathfrak{S}[d_1, \cdots, d_v]$, *then* $\mathfrak{q}' \cap \mathfrak{S}$ *contains some* $\mathfrak{q}_i$ *properly, provided that* $\mathfrak{S}$ *is Noetherian.*

Proof. Set $\mathfrak{q} = \mathfrak{q}' \cap \mathfrak{S}$. Since $\mathfrak{q}'$ is an imbedded prime divisor of $a\mathfrak{S}'$, $\mathfrak{S}'_{\mathfrak{q}'}$ is not a normal ring ([10, § 9]). Therefore $\mathfrak{S}_{\mathfrak{q}}$ is not normal because $\mathfrak{S}' \subseteq \mathfrak{o}'$. This shows that $\mathfrak{q}$ contains one of $\mathfrak{q}_i$, say $\mathfrak{q}_1$ by Lemma 1 ([10, § 9]). If $\mathfrak{q} = \mathfrak{q}_1$, we have a contradiction to the choice of $d_1, \cdots, d_v$ and our assertion is proved.

Now we proceed with the proof of Theorem 3. We choose elements $d_1, \cdots, d_v$ of $\mathfrak{o}'$ as in Lemma 2 with $\mathfrak{S} = \mathfrak{o}_2$; then repeat this process with $\mathfrak{S} = \mathfrak{o}_2[d_1, \cdots, d_v]$ and so on; the existence of such d_i's follows from the corollary to Theorem 2. Then by the finiteness of chains of prime ideals in $\mathfrak{o}_2$ (or by the finiteness of dimension over I of the field of quotients L of $\mathfrak{o}$), after a finite number of steps we reach a ring $\mathfrak{o}^*$ in which $a\mathfrak{o}^*$ has no imbedded prime divisor. Since $\mathfrak{o}_2[1/a] = \mathfrak{o}''$ is a normal ring, $\mathfrak{o}^*[1/a]$ is a normal ring. Therefore the fact that $a\mathfrak{o}^*$ has no imbedded prime divisor shows that $\mathfrak{o}^*$ is a normal ring (on account of the property proved in Lemma 1) ([10, § 9]). Therefore $\mathfrak{o}^* = \mathfrak{o}'$ and $\mathfrak{o}'$ is a finite $\mathfrak{o}$-module, because we added only a finite number of elements of $\mathfrak{o}'$ at each step.

6. Preliminaries from the theory of valuation rings.

LEMMA 1. *An integral domain* $\mathfrak{v}$ *is a valuation ring if and only if the set of all principal ideals of* $\mathfrak{v}$ *is linearly ordered by inclusion. Equivalently, if an element a of the field of quotients of* $\mathfrak{v}$ *is not in* $\mathfrak{v}$, *then* a^{-1} *is in* $\mathfrak{v}$. (Krull [4])

Since the proof is easy and is well known, we will omit it.

Remark. From this lemma, we see that if $\mathfrak{v}$ is a valuation ring and if $\mathfrak{p}$ is a prime ideal of $\mathfrak{v}$, then $\mathfrak{v}_{\mathfrak{p}}$ and $\mathfrak{v}/\mathfrak{p}$ are valuation rings. Further it is easy to see that $\mathfrak{p}\mathfrak{v}_{\mathfrak{p}} = \mathfrak{p}$ (set-theoretically).

LEMMA 2. *Let* $\mathfrak{a}$ *be an ideal of an integral domain* $\mathfrak{o}$. *Let L be a field containing* $\mathfrak{o}$. *If x is an element of L and if* $\mathfrak{a}\mathfrak{o}[x]$ *contains the identity, then* $\mathfrak{a}\mathfrak{o}[x^{-1}]$ *does not contain the identity.* (Chevalley; see Cohen-Seidenberg [3])

Proof. We may asume that $\mathfrak{a}$ is a prime ideal in $\mathfrak{o}$, for, if not, we may replace $\mathfrak{a}$ by any of its prime divisor. Similarly, considering $\mathfrak{o}_{\mathfrak{a}}$ instead of $\mathfrak{o}$, we may assume that $\mathfrak{a}$ is the unique maximal ideal of $\mathfrak{o}$. Since $1 \in \mathfrak{a}\mathfrak{o}[x]$,

there exists a relation $1 = p_0 + p_1 x + \cdots + p_n x^n$ with $p_i \varepsilon \mathfrak{a}$. Since $\mathfrak{a}$ is the unique maximal ideal of $\mathfrak{o}$, $1 - p_0$ is a unit in $\mathfrak{o}$ and x^{-1} is integral over $\mathfrak{o}$. Therefore $\mathfrak{a}\mathfrak{o}[x^{-1}] \neq \mathfrak{o}[x^{-1}]$ ([10, § 4]).

LEMMA 3. *Let $\mathfrak{o}$ be a subring of a field L and let $\mathfrak{p}$ be a prime ideal of $\mathfrak{o}$. Then there exists a valuation ring $\mathfrak{v}$ of L which dominates $\mathfrak{o}_{\mathfrak{p}}$.* (Krull [4])

Proof. Let F be the set of subrings $\mathfrak{S}$ of L which contain $\mathfrak{o}$ such that $\mathfrak{p}\mathfrak{S} \neq \mathfrak{S}$. Then the set F is an inductive set, because members $\mathfrak{S}$ of F are characterized by $1 \not\varepsilon \mathfrak{p}\mathfrak{S}$ and $\mathfrak{o} \subseteq \mathfrak{S} \subseteq L$. Therefore there exists a maximal member $\mathfrak{v}$ in F by Zorn's lemma. If x is an element of L which is not in $\mathfrak{v}$, $\mathfrak{p}\mathfrak{v}[x]$ contains the identity by the maximality of $\mathfrak{v}$. Therefore $1 \not\varepsilon \mathfrak{p}\mathfrak{v}[1/x]$ by Lemma 2 and we see that $1/x \varepsilon \mathfrak{v}$ again by the maximality of $\mathfrak{v}$. Therefore $\mathfrak{v}$ is a valuation ring by Lemma 1. The existence of $\mathfrak{v}$ which dominates $\mathfrak{o}_{\mathfrak{p}}$ is easily seen if we consider $\mathfrak{o}_{\mathfrak{p}}$ instead of $\mathfrak{o}$.

PROPOSITION 4. *Let $\mathfrak{v}$ be a valuation ring of a field L and let $\mathfrak{p}$ be the maximal ideal of $\mathfrak{v}$. Let $\mathfrak{o}$ be a valuation ring of the residue class field $\mathfrak{v}/\mathfrak{p}$. Then there exists a uniquely determined valuation ring $\mathfrak{v}'$ of L which contains $\mathfrak{p}$ as a prime ideal and satisfies $\mathfrak{v}'/\mathfrak{p} = \mathfrak{o}$. With this $\mathfrak{v}'$, we have $\mathfrak{v}'_{\mathfrak{p}} = \mathfrak{v}$.*

Proof. If there exists such $\mathfrak{v}'$, it must be the set of all elements of $\mathfrak{v}$ whose residue classes modulo $\mathfrak{p}$ are contained in $\mathfrak{o}$. Therefore we have only to show that this set $\mathfrak{v}'$ is a valuation ring, because if we see that $\mathfrak{v}'$ is a valuation ring then the last assertion is easy. Let a be an element of L which is not in $\mathfrak{v}'$. If a is not in $\mathfrak{v}$, then $1/a \varepsilon \mathfrak{p}$ and $1/a \varepsilon \mathfrak{v}'$. Assume that a is in $\mathfrak{v}$ and let a' be the residue class of a modulo $\mathfrak{p}$. Since $\mathfrak{o}$ is a valuation ring, $1/a'$ is in $\mathfrak{o}$ and $1/a \varepsilon \mathfrak{v}'$. Since obviously $\mathfrak{v}'$ is an integral domain, we see that $\mathfrak{v}'$ is a valuation ring.

We call $\mathfrak{v}'$ the *composite* of $\mathfrak{v}$ and $\mathfrak{o}$.

PROPOSITION 5. *Let $\mathfrak{o}$ be a subring of a field L and let $\mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \cdots \subset \mathfrak{p}_r$ be a chain of prime ideals $\mathfrak{p}_i$ in $\mathfrak{o}$. Then there exists a valuation ring $\mathfrak{v}$ of L which has prime ideals $\mathfrak{n}_1, \cdots, \mathfrak{n}_r$ such that $\mathfrak{p}_i = \mathfrak{n}_i \cap \mathfrak{o}$ for each i.* (Krull [4])

Proof. We prove our assertion by induction on r. Let $\mathfrak{v}_1$ be a valuation ring of L whose maximal ideal $\mathfrak{n}_1$ lies over $\mathfrak{p}_1$; existence of $\mathfrak{v}_1$ follows from Lemma 3. Then by our induction asumption, there exists a valuation ring $\mathfrak{v}'$ of $\mathfrak{v}_1/\mathfrak{n}_1$ which has prime ideals $\mathfrak{n}'_2, \cdots, \mathfrak{n}'_r$ such that $\mathfrak{n}'_i \cap (\mathfrak{o}/\mathfrak{p}_1)$

$=\mathfrak{p}_i/\mathfrak{p}_1$ for each i. Then by Proposition 4, we see the existence of a valuation ring with the required property.

Remark. The proof of Proposition 5 shows that if we are given a valuation ring $\mathfrak{v}_1$ which dominates $\mathfrak{o}_{\mathfrak{p}_1}$, then the valuation ring $\mathfrak{v}$ may be chosen as the composite of $\mathfrak{v}_1$ and a valuation ring of the residue class field of $\mathfrak{v}_1$.

PROPOSITION 6. *Let $\mathfrak{o}$ be a subring of a field L. If an element x of L is not integral over $\mathfrak{o}$, then there exists a valuation ring $\mathfrak{v}$ of L which contains $\mathfrak{o}$ and does not contain x.* (Krull [4])

Proof. If x is integral over $\mathfrak{o}[1/x]$, then x is integral over $\mathfrak{o}$. Therefore x is not integral over $\mathfrak{o}[1/x]$ and we may assume that $1/x \varepsilon \mathfrak{o}$. Let $\mathfrak{p}$ be a prime ideal of $\mathfrak{o}$ containing $1/x$ and let $\mathfrak{v}$ be a valuation ring of L which dominates $\mathfrak{o}_{\mathfrak{p}}$ (by Lemma 3). Then $x^{-1}\mathfrak{v} \neq \mathfrak{v}$ and $\mathfrak{v}$ does not contain x.

COROLLARY. *A normal ring $\mathfrak{o}$ is the intersection of all valuation rings (of its field of quotients) which contain $\mathfrak{o}$ (and conversely).* (Krull [4])

Chapter 2. The Notion of Model.

Throughout this chapter, we fix a ground ring I unless the contrary is explicitly stated. Further, all function fields will be assumed to be contained in some fixed field, unless the contrary is explicitly stated.

1. Places and correspondences. A valuation ring (or a field) which is a ring of quotients of the ground ring I is called a *ground place* (of I). When L is a function field (over I), a valuation ring $\mathfrak{v}$ of L is called a *place* of L (over I) if $\mathfrak{v}$ dominates some ground place (of I). Since we fix the ground ring I, we shall often omit the term "over I" or "of I".

When P and P' are spots, we say that P and P' *correspond* to each other or that P corresponds to P' if there exists a place which dominates both P and P'. (The function fields of P and P' may be distinct from each other.)

THEOREM 1. *A spot P corresponds to a spot P' if and only if the ideal $\mathfrak{a}$ of $P[P']$ which is generated by the maximal ideals $\mathfrak{m}$ and $\mathfrak{m}'$ of P and P' does not contain* 1. *In that case, if $\mathfrak{p}$ is a prime ideal of $P[P']$ which contains $\mathfrak{a}$, then $P[P']_{\mathfrak{p}}$ is a spot which domiantes P, P'. Conversely, if a spot Q dominates both P and P', then Q dominates a spot which is a ring of quotients of $P[P']$ obtained as above.*

Proof. Assume that P corresponds to P'. Then there exists a place $\mathfrak{v}$ which dominates both P and P'. Let $\mathfrak{w}$ be the maximal ideal of $\mathfrak{v}$. Then $\mathfrak{w}$ contains $\mathfrak{m}$ and $\mathfrak{m}'$, and therefore also $\mathfrak{a}$, which shows that $\mathfrak{a}$ does not contain 1. Conversely, assume that $\mathfrak{a}$ does not contain 1. Let $\mathfrak{p}$ be a prime ideal of $P[P']$ which contains $\mathfrak{a}$. Let $\mathfrak{o}$ and $\mathfrak{o}'$ be affine rings such that P and P' are rings of quotients of $\mathfrak{o}$ and $\mathfrak{o}'$ respectively. Then $P[P']$ is a ring of quotients of $\mathfrak{o}[\mathfrak{o}']$ and $\mathfrak{o}[\mathfrak{o}']$ is an affine ring. Therefore $P[P']_{\mathfrak{p}}$ is a spot. Since $\mathfrak{p}$ lies over $\mathfrak{m}$ and $\mathfrak{m}'$ (because $\mathfrak{m}$ and $\mathfrak{m}'$ are maximal ideals), the maximal ideal $\mathfrak{p}P[P']_{\mathfrak{p}}$ of $P[P']_{\mathfrak{p}}$ lies over $\mathfrak{m}$ and $\mathfrak{m}'$, which shows that $P[P']_{\mathfrak{p}}$ dominates P and P'. Next we assume that a spot Q dominates both P and P'. Let $\mathfrak{v}'$ be any place which dominates Q. Then $\mathfrak{v}'$ dominates both P and P'. Therefore P and P' correspond to each other. Let $\mathfrak{p}'$ be the prime ideal of $P[P']$ over which the maximal ideal of Q lies. Then Q dominates $P[P']_{\mathfrak{p}'}$ and this last spot dominates P and P'. Thus the proof is completed.

LEMMA 1. *If two spots P and P' are rings of quotients of the same ring $\mathfrak{o}$, then P cannot correspond to P' unless $P = P'$.*

Proof. If $P \neq P'$, then there exists an element a of $\mathfrak{o}$ which is a unit in one of the rings P and P' and a non-unit in another, and P cannot correspond to P'.

2. Definition of models. A set A of spots of a function field L is called an *affine model* (of L) if there exists an affine ring $\mathfrak{o}$ of L such that A is the set of all spots which are rings of quotients of $\mathfrak{o}$ (with respect to prime ideals). Such an affine ring $\mathfrak{o}$ is called an affine ring of A and A is called the affine model defined by $\mathfrak{o}$. A given affine model can be defined by only one affine ring. Indeed:

LEMMA 1. *If A is an affine model and if $\mathfrak{o}$ is an affine ring which defines A, then $\mathfrak{o}$ is the intersection of all spots in A.*

This follows from the remark at the end of [10, § 9].

Now we define the notion of models. A non-empty set M of spots of a function field L is called a *model* of L if M is the union of a finite number of affine models of L and if two different spots in M never correspond to each other.

Observe that an affine model is a model by virtue of Lemma 2.1.1.

When M is a model of a function field L, we say that L is the function field of M.

Let M be a model of a function field L. When $\mathfrak{v}$ is a place of a function

field L' containing L, there may be a spot P in M which is dominated by $\mathfrak{v}$; if so, then P is uniquely determined; it is called the *center* of $\mathfrak{v}$ on M. A model M of a function field L is said to be *complete* if every place of L has a center on M.

Remark 1. An affine model is complete if and only if its affine ring is integral over the ground ring.

The proof follows from Proposition 1.6.

Remark 2. A model M of a function field L is complete if and only if one of the following conditions is satisfied:

(1) Every place of a function field containing L has a center on M.

(2) Every spot of L corresponds to a spot in M.

(3) Every spot of a function field which contains L corresponds to a spot in M.

In order to prove this, we first establish

LEMMA 2. *Let L be a function field and let L' be a function field which contains L. Then*

1) *If $\mathfrak{v}'$ is a place of L', then $\mathfrak{v}' \cap L$ is a place of L.*

2) *If $\mathfrak{v}$ is a place of L, then there exists a place $\mathfrak{v}'$ of L' such that $\mathfrak{v} = \mathfrak{v}' \cap L$.*

3) *If P is a spot of L, then there exists a place $\mathfrak{v}$ of L which dominates P.*

This follows immediately from results in § 6 of Chapter 1.

Proof of Remark 2. Equivalence with (1) is immediate from Lemma 2. We will prove the equivalence with (2) or (3). Let L' be any function field containing L. Assume first that M is complete. Let P be any spot of L or L'. Then there exists a place $\mathfrak{v}'$ of L' which dominates P. Let Q be the center of $\mathfrak{v}'$ on M. Then P corresponds to Q. Conversely, assume that M is not complete. Then there exists a place $\mathfrak{v}$ of L or L' which has no center on M. Let $A_1, \cdots, A_n$ be affine models of L such that $M = \bigcup_i A_i$ and let $\mathfrak{o}_i$ be the affine ring of A_i. Since $\mathfrak{v}$ has no center on M, $\mathfrak{v}$ does not contain any of the rings $\mathfrak{o}_i$; let x_i be an element of $\mathfrak{o}_i$ which is not in $\mathfrak{v}$ and set $y_i = 1/x_i$. Further let $y_{n+1}, \cdots, y_r$ be elements of $\mathfrak{v}$ such that $I[y_1, \cdots, y_r] = \mathfrak{S}$ is an affine ring of L or L'. Set $P = \mathfrak{S}_{(\mathfrak{m} \cap \mathfrak{S})}$, where $\mathfrak{m}$ is the maximal ideal of $\mathfrak{v}$. Let Q be any spot in M. Then Q is a ring of quotients of some $\mathfrak{o}_i$, say $\mathfrak{o}_1$.

Then x_1 is in Q and $y_1 = 1/x_1$ is a non-unit in P. Therefore y_1 is a unit in the ring $P[Q]$, which shows that P does not correspond to Q by Theorem 1. Thus we see that this spot P does not correspond to any spot in M and the proof is completed.

THEOREM 2. *Let $x_0 = 1, x_1, \cdots, x_n$ be elements of a function field L such that $I[x_1, \cdots, x_n]$ is an affine ring of L. Let A_i be the affine model defined by $\mathfrak{o}_i = I[x_0/x_i, \cdots, x_n/x_i]$ for each i such that $x_i \neq 0$. Then the union M of all A_i is a complete model of L.*

This model M is called the *projective model* of L defined by the *affine coordinates* $(x_1, \cdots, x_n)$. (A model is called a projective model if it is a projective model defined by a suitable affine coordinates.)

Proof. We first show that M is a model. Assume the contrary. Then there exist two different spots P and P' in M which correspond to each other. By Lemma 2.1.1, P and P' cannot be in the same affine model A_i. Since $(x_j/x_i)/(x_k/x_i) = x_j/x_k$, we may assume that P is in A_0 and P' is in A_1. Set $\mathfrak{o} = \mathfrak{o}_0[\mathfrak{o}_1]$. Then by Theorem 1, there exists a prime ideal $\mathfrak{p}$ of $\mathfrak{o}$ such that $\mathfrak{o}_\mathfrak{p}$ dominates both P and P'. Since $\mathfrak{o}$ contains both x_1 and $1/x_1$, x_1 is a unit in $\mathfrak{o}$. Since P contains x_1 and since $\mathfrak{o}_\mathfrak{p}$ dominates P, we see that x_1 is a unit in P and P contains $\mathfrak{o}$. It follows that $\mathfrak{o}_\mathfrak{p} = P$ (because P is a ring of quotients of $\mathfrak{o}_0$; see [10, § 2]). Similarly we see that $\mathfrak{o}_\mathfrak{p} = P'$, whence $P = P'$, which is a contradiction. Thus we have proved that M is a model. Now let $\mathfrak{v}$ be a place of L and let v be a valuation of L defined by $\mathfrak{v}$. We choose x_i such that $v(x_i) \leqq v(x_j)$ for any $j = 0, \cdots, n$. Then $\mathfrak{o}_i$ is contained in $\mathfrak{v}$. Let $\mathfrak{q}$ be the intersection of the maximal ideal of $\mathfrak{v}$ with $\mathfrak{o}_i$. Then $Q = (\mathfrak{o}_i)_\mathfrak{q}$ is dominated by $\mathfrak{v}$. Therefore Q is the center of $\mathfrak{v}$ on M, which shows that M is complete.

The notion of projective model can be defined by *homogeneous coordinates* as follows: Let $z_0, \cdots, z_n$ (at least one of z_i, say z_0, is not zero) be elements of a certain field containing I. Let A_i be the affine model defined by the affine ring $I[z_0/z_i, \cdots, z_n/z_i]$ for each i such that $z_i \neq 0$. Then the union of all A_i is the projective model defined by the affine coordinates $(z_1/z_0, \cdots, z_n/z_0)$. This model is called the projective model defined by the homogeneous coordinates $(z_0, \cdots, z_n)$. Observe that the projective model defined by the affine coordinates $(x_1, \cdots, x_n)$ is also defined by the homogeneous coordinates $(t, x_1t, \cdots, x_nt)$ with an arbitrary nonzero element t.

Let M be a projective model of a function field L. An affine ring $\mathfrak{o}$ is called a homogeneous coordinate ring of M if $\mathfrak{o}$ is generated by a system of homogeneous coordinates over the ground ring I and if it contains an

algebraically independent element over the function field L of M; when $\mathfrak{o}$ is expressed in the form $I[z_0, \cdots, z_n]$, then we understand that $(z_0, \cdots, z_n)$ is a system of homogeneous coordinates which defines M, unless the contrary is explicitly stated. Observe that one nonzero z_i is transcendental over L if and only if every nonzero z_j is transcendental over L. Let K be the field of quotients of $\mathfrak{o}$ and asume that z_0 is transcendental over L. Then $K = L(z_0)$. An element a of K is called homogeneous if there exists an element b of L such that $a = bz_0{}^r$ with a rational integer r; this r is called the degree of homogeneity (or merely degree) of a.

3. Specializations. A quasi-local ring P is said to be a *specialization* of another quasi-local ring P' if P' is a ring of quotients of P (necessarily with respect to a prime ideal). Observe that if a valuation ring $\mathfrak{v}$ is the composite of a valuation ring $\mathfrak{v}'$ and a valuation ring of the residue class field of $\mathfrak{v}'$, then $\mathfrak{v}$ is a specialization of $\mathfrak{v}'$.

PROPOSITION 1. *If a spot P' is a specialization of another spot P and if P' is in a model M, then P is also in M; if $P \neq P'$, then* $\dim P' < \dim P$.

Proof. Since there exists an affine model A such that $P' \varepsilon A \subseteq M$, we may assume that M is an affine model. Let $\mathfrak{o}$ be the affine ring of M and let $\mathfrak{p}'$ be the prime ideal of $\mathfrak{o}$ such that $P' = \mathfrak{o}_{\mathfrak{p}'}$. Let $\mathfrak{m}$ be the prime ideal of P' such that $P = P'_{\mathfrak{m}}$. Then $P = \mathfrak{o}_{(\mathfrak{m} \cap \mathfrak{o})}$, which shows that P is in M. If $P \neq P'$, then $\operatorname{rank} P < \operatorname{rank} P'$, whence $\dim P > \dim P'$ by Corollary 1 to Theorem 1.1.

PROPOSITION 2. *Assume that a spot P' of a function field L is a specialiaztion of another spot P (necessarily of L). If* $\dim P' < \dim P$, *then there exists a spot P^* of L such that* 1) *P' is a specialization of P^*,* 2) *P^* is a specialization of P and* 3) $\dim P^* = \dim P' + 1$.

The proof is easy making use of Corollary 1 to Theorem 1.1.

PROPOSITION 3. *Assume that the ground ring I is either a field or a Dedekind domain which has infinitely many prime ideal. If M is a model and if P is a spot in M, then there exists a spot P' in M which is a specialization of P and of dimension* 0.

Proof. We may assume that M is an affine model. Then the proof is immediate from Corollary 6 to Proposition 1.1.

PROPOSITION 4. *Let M be a model of a function field L and let L' be a function field which contains L. Assume that spots P and P' in M are*

dominated by places $\mathfrak{v}$ *and* $\mathfrak{v}'$ *of* L' *respectively and that* $\mathfrak{v}'$ *is a specialization of* $\mathfrak{v}$. *Then* P' *is a specialization of* P. *Conversely, if a spot* Q' *of* L *is a specialization of another spot* Q *and if a place* $\mathfrak{w}$ *of* L' *dominates* Q, *then there exists a place* $\mathfrak{w}'$ *of* L' *which is a specialization of* $\mathfrak{w}$ *and dominates* Q'.

Proof. Let A be an affine model of L which contains P' and is contained in M. Let $\mathfrak{o}$ be the affine ring of A and let $\mathfrak{m}$ and $\mathfrak{m}'$ be the maximal ideals of $\mathfrak{v}$ and $\mathfrak{v}'$ respectively. Then we have $\mathfrak{o} \subseteq P' \subseteq \mathfrak{v}' \subseteq \mathfrak{v}$. Therefore we can set $P^* = \mathfrak{o}_{(\mathfrak{m} \cap \mathfrak{o})}$ and $P^{**} = \mathfrak{o}_{(\mathfrak{m}' \cap \mathfrak{o})}$. Then P^* and P^{**} are the centers of $\mathfrak{v}$ and $\mathfrak{v}'$ on A, hence on M. Therefore $P^* = P$ and $P^{**} = P'$ and P' is a specialization of P because $\mathfrak{m} \subseteq \mathfrak{m}'$. The converse follows from the remark after Proposition 1.5.

4. Joins of models. Let M and M' be models (of function field L and L' respectively. We say that M *dominates* M' (in symbols $M' \leqq M$) if every spot in M dominates some spot in M'.

If P and P' are spots, the set of spots which are rings of quotients of $P[P']$ and dominate both P and P' is called the *join* of P and P'; it will be denoted by $J(P, P')$.

Remark. $J(P, P')$ is empty if and only if P does not correspond to P', by virtue of Theorem 1.

Now, for two sets M and M' of spots, the union of all $J(P, P')$, where P and P' run over all spots in M and M' respectively, is called the *join* of M and M' and will be denoted by $J(M, M')$.

THEOREM 3. *Let* M *and* M' *be models of function fields* L *and* L' *respectively. Then the join* $J(M, M')$ *of* M *and* M' *is a model of* $L(L')$. *Further* $J(M, M')$ *dominates both* M *and* M' *and if a model* M'' *(of a function field which contains* $L(L')$*) dominates* M *and* M', *it dominates* $J(M, M')$.

Proof. Our assertion, exception for the fact that $J(M, M')$ is a model, follows from Theorem 1. We prove that $J(M, M')$ is a model. Assume that spots P''_1 and P''_2 in $J(M, M')$ correspond to each other. We take spots $P_1, P_2 \varepsilon M$ and $P'_1, P'_2 \varepsilon M'$ such that $P''_i \varepsilon J(P_i, P'_i)$ for each i. Let $\mathfrak{v}$ be a place which dominates both P''_1 and P''_2. Then $\mathfrak{v}$ dominates P_1, P_2, P'_1 and P'_2. It follows that these spots correspond to each other. Therefore by the definition of models, we have $P_1 = P_2$ and $P'_1 = P'_2$. Thus we see that P''_1 and P''_2 are rings of quotients of the same ring; by Lemma 2.1.1, we have $P''_1 = P''_2$. Now we have only to show that $J(M, M')$ is the union of a finite number of affine models. In order to do this, we have only to prove

LEMMA 1. *If A and A' are affine models, then the join $J(A,A')$ is an affine model.*

Proof. Let $\mathfrak{o}$ and $\mathfrak{o}'$ be the affine rings of A and A' respectively. Then the ring $\mathfrak{o}''=\mathfrak{o}[\mathfrak{o}']$ is also an affine ring. Let A'' be the affine model defined by $\mathfrak{o}''$. By the definition of $J(A,A')$, every member of $J(A,A')$ is a ring of quotients of $\mathfrak{o}''$, whence $J(A,A')\subseteq A''$. Conversely, let P'' be a spot in A'' and let $\mathfrak{p}''$ be the prime ideal of $\mathfrak{o}''$ such that $P''=\mathfrak{o}''_{\mathfrak{p}''}$. Then P'' contains spots $P=\mathfrak{o}_{(\mathfrak{p}''\cap\mathfrak{o})}$ and $P'=\mathfrak{o}_{(\mathfrak{p}''\cap\mathfrak{o}')}$. Therefore P'' is a ring of quotients of $P[P']$ ([10, § 2]) and dominates P and P'. Therefore P'' is in $J(A,A')$ and $A''\subseteq J(A,A')$. Thus the assertion is proved.

PROPOSITION 5. *If M and M' are complete models of function fields L and L' respectively, then $J(M,M')$ is a complete model of $L(L')$. If M and M' are projective models, then so is $J(M,M')$.*

Proof. Let $\mathfrak{v}$ be a place of $L(L')$. Then $\mathfrak{v}$ has centers P and P' on M and M' respectively. Let $\mathfrak{m}$ be the maximal ideal of $\mathfrak{v}$ and set $\mathfrak{n}=\mathfrak{m}\cap P[P']$. Then $P''=P[P']_{\mathfrak{n}}$, which is in $J(M,M')$ by Theorem 1, is dominated by $\mathfrak{v}$, which shows that $\mathfrak{v}$ has a center P'' on $J(M,M')$. It follows that $J(M,M')$ is a complete model. Now we assume that M and M' are projective models defined by homogeneous coordinates $(x_0,\cdots,x_m)$ and $(y_0,\cdots,y_n)$ respectively. We may assume here that no x_i and no y_j is zero. Let M'' be the projective model defined by the homogeneous coordinates $(x_iy_j; i=0,\cdots,m; j=0,\cdots,n)$. We denote by $\mathfrak{o}_i$ and $\mathfrak{o}'_j$ the affine rings $I[x_0/x_i,\cdots,x_m/x_i]$ and $I[y_0/y_j,\cdots,y_n/y_j]$ respectively. Let A_i and A'_j be the affine models defined by $\mathfrak{o}_i$ and $\mathfrak{o}'_j$ respectively. On the other hand, let $\mathfrak{o}''_{ij}$ be the affine ring $\mathfrak{o}_i[\mathfrak{o}'_j]$ and let A''_{ij} be the affine model defined by $\mathfrak{o}''_{ij}$. Then $\mathfrak{o}''_{ij}$ is generated by the elements x_ry_s/x_iy_j (i and j are fixed; $0\leqq r\leqq m$, $0\leqq s\leqq n$) over I. Therefore M'' is the union of all A''_{ij}. Since $\mathfrak{o}''_{ij}=\mathfrak{o}_i[\mathfrak{o}'_j]$, $A''_{ij}=J(A_i,A'_j)$. Therefore M'' is the join $J(M,M')$ of M and M'. Thus we see that $J(M,M')$ is a projective model.

If a model M dominates another model M', then $J(M,M')=M$. From this fact we deduce

LEMMA 2. *Assume that a model M of a function field L dominates a model M' of a function field L'. Then there exist affine models A_{ij} of L and affine models A'_j of L' $(j=1,\cdots,m; i=1,\cdots,n(j))$ such that* 1) $A_{ij}\geqq A'_j$ *and* 2) $M=\bigcup_{ij}A_{ij}$, $M'=\bigcup_j A'_j$.

LEMMA 3. *Let M and M' be models of the same function field L.*

Assume that there exists a model M'' of L which contains both M and M', then $M \cap M' = J(M, M')$.

The proof is easy.

5. Derived normal model of a model. We say that a model M is *normal* if every spot in M is normal.

NOTATION. *Let P be a spot of a function field L and let L' be a finite algebraic extension of L. Further let $\mathfrak{o}$ be the integral closure of P in L'. Then we denote by $N(P;L')$ the set of spots which are rings of quotients of $\mathfrak{o}$ with respect to maximal ideals of $\mathfrak{o}$. When M is a set of spots of L, then the union of all $N(P;L')$, where P runs over all spots in M, will be denoted by $N(M;L')$. On the other hand, $N(P;L)$ and $N(M;L)$ are denoted by $N(P)$ and $N(M)$ respectively.*

Remark 1. $N(P;L')$ is a finite set, because $\mathfrak{o}$ is a finite P-module.[3]

Remark 2. In §§ 1-4, our assumption that ground rings satisfy the finiteness condition for integral extensions did not play any rôle. In the present section this condition will be used in an esential manner. Without it, it would not be true in general that the derived normal ring of a spot P is a finite P-module. (If the function field is separably generated over the ground ring I, then the derived normal ring of P is a finite P-module; cf. the appendix to the second paper of this series; in this case, assuming further that L' is separably generated, the results of the present section hold without assuming the finiteness condition.)

THEOREM 4. *Let M be a model of a function field L and let L' be a finite algebraic extension of L. Then $N(M;L')$ is a normal model. If a normal model M' of a function field containing L' dominates M, then M' dominates $N(M;L')$.*

This model $N(M;L')$ is called the *derived normal model* of M in L'; $N(M)$ is called the derived normal model of M.

Proof. We first prove the first assertion in the case where M is an affine model. Let $\mathfrak{o}$ be the affine ring of M and let $\mathfrak{o}'$ be the integral closure of $\mathfrak{o}$ in L'. Then by Theorem 1.3, $\mathfrak{o}'$ is also an affine ring. On the other hand, $N(M;L')$ is the set of spots which are rings of quotients of $\mathfrak{o}'$, which shows

[3] It is well known that if $\mathfrak{o}$ is a (Noetherian) semi-local integral domain, then the derived normal ring of $\mathfrak{o}$ has only a finite number of maximal ideals (but it may not be Noetherian).

that $N(M;L')$ is an affine model and is normal. Now we consider the general case. By the above statement, we see that $N(M;L')$ is the union of a finite number of affine models. Assume that a spot $P^* \varepsilon N(M;L')$ corresponds to a spot $P^{**} \varepsilon N(M;L')$. Let P and P' be spots in M such that $P^* \varepsilon N(P;L')$ and $P^{**} \varepsilon N(P';L')$. Let $\mathfrak{v}$ be a place which dominates both P^* and P^{**}. Since P^* and P^{**} dominate P and P' respectively, $\mathfrak{v}$ dominates both P and P', which shows that $P = P'$. Therefore P^* and P^{**} are rings of quotients of the same ring (which is the integral closure of P in L'). Therefore by Lemma 2.1.1, we have $P^* = P^{**}$. Thus we see that $N(M;L')$ is a model and is obviously normal. Now we prove the last assertion. Let P'' be a spot in M'. Then there exists a spot P in M which is dominated by P''. Since P'' is a normal ring and since the field of quotients of P'' contains L', P'' contains the integral closure $\mathfrak{o}'$ of P in L'. Let $\mathfrak{m}''$ be the maximal ideal of P''. Since P'' dominates P, $\mathfrak{m}'' \cap P$ is the maximal ideal of P. Therefore $\mathfrak{m}'' \cap \mathfrak{o}'$ lies over the maximal ideal of P, which shows that $\mathfrak{m}'' \cap \mathfrak{o}'$ is a maximal ideal of $\mathfrak{o}'$ ([10, § 4]). Therefore the ring $\mathfrak{o}'_{(\mathfrak{m}'' \cap \mathfrak{o}')}$ is in $N(M;L')$ and is dominated by P''. This completes the proof.

THEOREM 5. *Let M be a model of a function field L and let L' be a finite algebraic extension of L. If M is a complete model, then $N(M;L')$ is also complete. If M is a projective model, then $N(M;L')$ is also a projective model.*

Proof. Since the first half is easy, we prove only the last assertion. We assume that M is the projective model defined by the homogeneous coordinates $(y_0, \cdots, y_n)$, where we may assume that y_0 is transcendental over L and that all y_i are not zero. Set $\mathfrak{h} = I[y_0, \cdots, y_n]$. Since $\mathfrak{h}$ is an affine ring, the integral closure $\mathfrak{h}'$ of $\mathfrak{h}$ in $L'(y_0)$ is a finite $\mathfrak{h}$-module. Let $\mathfrak{i}$ be the set of elements of $\mathfrak{h}'$ which are homogeneous and of positive degree (where the notion of homogeneity and of degree are defined in the same way as in the case of $L(y_0)$). Since $\mathfrak{h}'$ is a finite $\mathfrak{h}$-module, $\mathfrak{h}'' = \mathfrak{h}[\mathfrak{i}]$ is a finite $\mathfrak{h}$-module. Let r be a natural number and let $w_0, \cdots, w_t$ be a set of generators of the I-module of all elements of $\mathfrak{h}''$ of degree r; here we choose w_i to be y_i^r for $i \leqq n$. Further, let $M(r)$ be the projective model defined by the homogeneous coordinates $(w_0, \cdots, w_t)$ and let $L(r)$ be the function field of $M(r)$. It is obvious that $L(r)$ is contained in L'. We set $z_{ij} = w_i/w_j$ and $\mathfrak{o}_i = I[z_{0i}, \cdots, z_{ti}]$. Then

(1) For each $i \leqq n$, $\mathfrak{o}_i$ is integral over $I[y_0/y_i, \cdots, y_n/y_i]$.

Proof. Since w_j is integral over $\mathfrak{h}$, there exists a relation $w_j^u + c_1 w_j^{u-1}$

$+\cdots+c_u=0$ with $c_k \varepsilon \mathfrak{h}$; here we may assume that each c_k is a homogeneous polynomial in $y_0,\cdots,y_n$ of degree rk. Then c_k/y_i^{rk} is in $I[y_0/y_i,\cdots,y_n/y_i]$ and w_j/y_i^r is integral over $I[y_0/y_i,\cdots,y_n/y_i]$.

(2) For sufficiently large r and for $i \leqq n$, $\mathfrak{o}_i$ is the integral closure of $I[y_0/y_i,\cdots,y_n/y_i]$ in L'.

Proof. Let c be any element of the integral closure $\mathfrak{o}'_i$ of $I[y_0/y_i,\cdots,y_n/y_i]$ in L'. Then there exists one integer d such that cy_i^d is integral over $\mathfrak{h}$. If r is chosen to be not less than d, then, since cy_i^r is also in $\mathfrak{h}''$ in this case, we see that c is in $\mathfrak{o}_i$. Since $\mathfrak{o}'_i$ is a finite $I[y_0/y_i,\cdots,y_n/y_i]$-module, we see that for sufficiently large r, $\mathfrak{o}_i$ contains $\mathfrak{o}'_i$. Now, by virtue of the observation in (1) above, we see that $\mathfrak{o}_i=\mathfrak{o}'_i$ for sufficiently large r.

The fact proved in (2) shows that $N(M;L')$ is a subset of $M(r)$ for sufficiently large r. Since $N(M;L')$ is a complete model, it follows that $N(M;L')=M(r)$ and $N(M;L')$ is a projective model of L'. Thus the assertion is proved.

6. Irreducible sets. When M is a model and when P is a spot in M, the set of spots in M which are specialization of P is called the *locus* of P in M and will be denoted by $M(P)$.

A subset E of a model M is called an *irreducible set* if there exists a spot P in M which satisfies the following two conditions: (1) E is contained in the locus of P in M and (2) for any finite number of spots $P_1,\cdots,P_n$ in M which are specializations of P and which are different from P, E is not contained in the union of the loci $M(P_i)$ of the spots P_i. We call P the *generating spot* of E; it is unique as we now show.

Assume that P' is also a generating spot of E. Let Q be a spot in E and let A be an affine model containing Q and contained in M. By Proposition 1 P and P' are in A. Let $\mathfrak{o}$ be the affine ring of A and let $\mathfrak{p}$ and $\mathfrak{p}'$ be the prime ideals of $\mathfrak{o}$ such that $P=\mathfrak{o}_{\mathfrak{p}}$, $P'=\mathfrak{o}_{\mathfrak{p}'}$. Let $\mathfrak{p}_1,\cdots,\mathfrak{p}_r$ be the set of all minimal prime divisors of $\mathfrak{p}+\mathfrak{p}'$ and set $P_i=\mathfrak{o}_{\mathfrak{p}_i}$. Since Q is a specialization of both P and P', the prime ideal $\mathfrak{q}$ such that $Q=\mathfrak{o}_{\mathfrak{q}}$ contains $\mathfrak{p}+\mathfrak{p}'$. Therefore Q is a specialization of some P_i. This shows that $E\cap A$ is contained in the union of the loci of the spots $P_1,\cdots,P_r$. Since the same holds for any affine model contained in M and since M is the union of a finite number of affine models, we have a contradiction if $P\neq P'$.

Remark. Since the notion of specialization does not depend on the choice of the model which carries the given spots, the notion of irreducible set does not depend on the choice of the model which carries the given sets.

LEMMA 1. *Let A be the affine model of a function field L defined by an affine ring $\mathfrak{o}$. Let E be a subset of A. Then E is irreducible if and only if the intersection $\mathfrak{p}$ of all prime ideals $\mathfrak{q}$ such that $\mathfrak{o}_{\mathfrak{q}}$ is in E is a prime ideal. In this case, $\mathfrak{o}_{\mathfrak{p}}$ is the generating spot of E.*

Proof. Assume first that $\mathfrak{p}$ is a prime ideal. If $\mathfrak{o}_{\mathfrak{q}}$ is in E ($\mathfrak{q}$ being a prime ideal of $\mathfrak{o}$), $\mathfrak{q}$ contains $\mathfrak{p}$ and $\mathfrak{o}_{\mathfrak{q}}$ is a specialization of the spot $P=\mathfrak{o}_{\mathfrak{p}}$. If there exist spots $\mathfrak{o}_{\mathfrak{p}_1},\cdots,\mathfrak{o}_{\mathfrak{p}_n}$ ($\mathfrak{p}_i$ being prime ideals of $\mathfrak{o}$) which are specializations of P and are different from P such that E is contained in the union of $A(\mathfrak{o}_{\mathfrak{p}_i})$, then by the definition of $\mathfrak{p}$, the intersection of all $\mathfrak{p}_i$ must be contained in $\mathfrak{p}$, which is a contradiction because $\mathfrak{p}\subset\mathfrak{p}_i$ ($\mathfrak{p}\neq\mathfrak{p}_i$). This shows that P is the generating spot of E. Conversely, assume that $\mathfrak{p}$ is not prime. Since $\mathfrak{p}$ is an intersection of prime ideals, $\mathfrak{p}$ is a semi-prime ideal. Therefore $\mathfrak{p}$ is the intersection of the minimal prime divisors $\mathfrak{p}_1,\cdots,\mathfrak{p}_r$ ($r>1$) of $\mathfrak{p}$ ([10, § 1]). Set $P_i=\mathfrak{o}_{\mathfrak{p}_i}$. If P is a spot whose locus contain E, then the prime ideal $\mathfrak{q}$ such that $P=\mathfrak{o}_{\mathfrak{q}}$ must be contained in $\mathfrak{p}$. Since E is contained in the union of the loci $P_1,\cdots,P_r$, P is not a generating spot of E, which shows that E is not irreducible. Thus the proof is completed.

On the other hand, with the same notations as above, if we set $E_i=E\cap A(P_i)$ for each i, then E_i is an irreducible set with generating spot P_i, which shows that E is the union of the irreducible sets E_i. Since a model is the union of a finite number of affine models, we have

LEMMA 2. *Every subset E of a model M is the union of a finite number of irreducible sets.*

COROLLARY. *If a subset E of a model M contains infinitely many spots, then there exists an irreducible subset of E which contains infinitely many spots.*

Again by the fact that a model is the union of a finite number of affine models, we see easily

LEMMA 3. *Let E be a subset of a model M of a function field L. Then a spot $P\in M$ is the generating spot of E (and consequently E is irreducible) if and only if* 1) *E is contained in the locus of P in M and* 2) *there exists an affine model A of L which is contained in M such that P is the generating spot of $E\cap A$.*

Remark. Let E be a subset of a model M. Then there may be different expressions of E as the union of a finite number of irreducible sets. But the following fact holds good.

There are irreducible sets $E_1, \cdots, E_n$ which satisfy the following three conditions; these E_i are uniquely determined:

1) E is the union of all E_i. 2) Every irreducible subset of E is contained in at least one of E_i. 3) There is no inclusion relation among the E_i.

Proof. Let $F_1, \cdots, F_r$ be irreducible subsets of E such that E is the union of them. Let P_i be the generating spot of F_i and set $E_i = E \cap M(P_i)$. We may assume without loss of generality that there is no inclusion relation among $E_1, \cdots, E_n$ and that each of $E_{n+1}, \cdots, E_r$ is contained in some E_i $(i \leqq n)$. These $E_1, \cdots, E_n$ satisfy the conditions 1) and 3). Let F be an irreducible subset of W with generating spot Q. Let A be an affine model contained in M such that $F \cap A$ is an irreducible set with generating spot Q (by Lemma 3). Then by the proof of Lemma 1, we see that Q is a specialization of some P_i, which shows that F is contained in some E_i. Therefore these E_i satisfy the condition 2). If $E'_1, \cdots, E'_m$ satisfy the above three conditions, then each E_i must be contained in some E'_j and E'_j must be contained in some E_k. Therefore the systems $E_1, \cdots, E_p$ and $E'_1, \cdots, E'_n$ must coincide with each other.

7. Zariski topology. Let M be a model. Let $\mathfrak{F}$ be the family of subsets F of M which satisfy the following conditions: (1) If a spot P is in F, then F contains the locus of P in M and (2) if F contains an irreducible set E, then F contains the generating spot of E.

This $\mathfrak{F}$ can be used as the family of closed sets in a topology, called the *Zariski topology* in M.

THEOREM 6. *A subset F of a model M is a closed set of M if and only if there exist a finite number of spots $P_1, \cdots, P_n$ in M such that F is the union of the loci $M(P_i)$.*

Proof. Each $M(P_i)$ is a closed set and therefore $\bigcup_i M(P_i)$ is also a closed set. Assume that F is a closed set of M. Let F' be the set of spots P in F which are not specialization of any other spots in F. We have only to show that F' is a finite set, which follows immediately from Lemma 2.6.2.

PROPOSITION 6. *A subset F of a model M is a closed set of M if and only if the following two conditions are satisfied: (1) If P is a spot in F, then $M(P) \subseteq F$ and (2) if a spot $P' \varepsilon M$ is not in F, then there exist a finite number of spots $P_1, \cdots, P_n \varepsilon M(P')$ which are different from P' such that $F \cap M(P') \subseteq \bigcup_i M(P_i)$.*

Proof. Assume first that F is closed. By the definition of closed sets, condition (1) must hold. Since F is a closed set, $F \cap M(P')$ is also a closed set. Therefore by Theorem 6, condition (2) holds also. We shall now prove the converse part. Let F' be defined as in the proof of Theorem 6. Then we have only to show that F' is a finite set. Assume the contrary. Then there exists an irreducible set F'' which contains infinitely many spots and is contained in F' by the corollary to Lemma 2.6.2. Let P'' be the generating spot of F''. Then by our assumption, P'' is not in F and therefore there exist a finite number of spots $P_1, \cdots, P_n \varepsilon M(P'')$ $(P_i \neq P'')$ such that F'' is contained in $\bigcup_i M(P_i)$ by condition (2), which is a contradiction to our assumption that F'' is irreducible. Thus the proof is completed.

THEOREM 7. *Let A be the affine model defined by an affine ring $\mathfrak{o}$. Then a subset F of A is a closed set of A if and only if there exists an ideal $\mathfrak{a}$ of $\mathfrak{o}$ such that F is the set of spots which are rings of quotients of $\mathfrak{o}$ with respect to prime ideals which contain $\mathfrak{a}$.*

In this case, we say that F is the *closed set of A defined by the ideal $\mathfrak{a}$* and that $\mathfrak{a}$ is an ideal of $\mathfrak{o}$ which defines F.

Proof. If F is a closed set, then by Theorem 6, there exist a finite number of spots $P_1, \cdots, P_n$ such that $F = \bigcup_i A(P_i)$. Let $\mathfrak{p}_i$ be the prime ideal of $\mathfrak{o}$ such that $P_i = \mathfrak{o}_{\mathfrak{p}_i}$. Then $\mathfrak{a} = \bigcap_i \mathfrak{p}_i$ has the required property. Conversely, asume that $\mathfrak{a}$ is an ideal of $\mathfrak{o}$. Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$ be the minimal prime divisors of $\mathfrak{a}$. Then a prime ideal $\mathfrak{p}$ of $\mathfrak{o}$ contains $\mathfrak{a}$ if and only if $\mathfrak{p}$ contains one of $\mathfrak{p}_i$'s. It follows that the set of rings of quotients of $\mathfrak{o}$ with respect to prime ideals containing $\mathfrak{a}$ is a closed set. Thus the proof is completed.

THEOREM 8. *Let M be a model of a function field L and let $A_1, \cdots, A_n$ be models of L such that M is the union of them. Then a subset F of M is closed if $F \cap A_i$ is a closed set of A_i for each i. Conversely, if F is a closed set of M, then $F \cap A$ is a closed set of A for any model A of L contained in M.*

Proof. Assume that $F \cap A_i$ is closed in A_i for each i. Then there exist a finite number of spots P_{ij} $(i = 1, \cdots, n; j = 1, \cdots, m(i))$ such that $F \cap A_i = \bigcup_j A_i(P_{ij})$. Then $F = \bigcup_{ij} A_i(P_{ij}) \subseteq \bigcup_{ij} M(P_{ij})$. We denote this last set by F'. Let P' be a spot in F'. Let i be such that P' is in A_i. Then $P' \varepsilon \bigcup_j A_i(P_{ij})$, which shows that $F = F'$ and we see that F is a closed set of M. Conversely, assume that F is a closed set of M. Take spots $P_1, \cdots, P_n \varepsilon F$ such that $F = \bigcup_i M(P_i)$. If $M(P_i)$ meets A, then P_i is in

A by Proposition 1 and in this case $M(P_i) \cap A$ coincides with $A(P_i)$. Therefore we see that $F \cap A$ is a closed set of A.

COROLLARY. *Let M' be a model of a function field L. If M' is contained in a model M of L, then M' is a subspace of M.*

Proof. If F is a closed set of M, then $F \cap M'$ is a closed set of M' by Theorem 8. Assume that F' is a closed set of M'. Take spots $P_1, \cdots, P_n \varepsilon F'$ such that $F' = \bigcup_i M'(P_i)$ and set $F = \bigcup_i M(P_i)$. Then F is a closed set of M and $F \cap M' = F'$. Thus the corollary is proved.

THEOREM 9. *Let M be a model of a function field L. A non-empty subset M' of M is again a model of L if and only if M' is an open set of M.*

Proof. Assume that M' is a model. We may assume that M is an affine model; because for every affine model A contained in M, $M' \cap A = J(M', A)$ (by Lemma 2.4.3) is a model and therefore, if our theorem is proved for $M = A$, then Theorem 8 asserts that M' is open. Further, we may assume that M' is an affine model, because M' is the union of a finite number of affine models. Let $\mathfrak{o}$ and $\mathfrak{o}'$ be affine rings of M and M' respectively and let F be the complements of M' in M. We have only to show that F is a closed set of M. Since M contains M', $\mathfrak{o}'$ contains $\mathfrak{o}$. Let $a_1, \cdots, a_n$ be elements of $\mathfrak{o}'$ such that $\mathfrak{o}' = \mathfrak{o}[a_1, \cdots, a_n]$ and set $\mathfrak{a}_i = \{a; a \varepsilon \mathfrak{o}, aa_i \varepsilon \mathfrak{o}\}$. Then each $\mathfrak{a}_i$ is an ideal of $\mathfrak{o}$. Set $\mathfrak{a} = \bigcap_i \mathfrak{a}_i$ and let F' be the closed set of M defined by $\mathfrak{a}$. Then it is sufficient to show that $F = F'$. Let $\mathfrak{p}$ be a prime ideal of $\mathfrak{o}$. Then $\mathfrak{o}_{\mathfrak{p}} \varepsilon F'$ if and only if $\mathfrak{a} \subseteq \mathfrak{p}$. Equivalently, $\mathfrak{p}$ contains some $\mathfrak{a}_i$, which means $\mathfrak{o}_{\mathfrak{p}}$ does not contain a_i, i.e., $\mathfrak{o}_{\mathfrak{p}}$ is not in M', $\mathfrak{o}_{\mathfrak{p}}$ is in F. Thus we see that $F = F'$. Now we shall prove the converse part of our theorem. Assume that M' is an open set of M. Let A_i $(i = 1, \cdots, n)$ be affine models such that M is the union of them. Then each A_i is an open set by the above proof. Therefore $M'_i = M' \cap A_i$ is an open set of M and therefore of A_i. If we can show that M'_i is a model, it is the union of a finite number of affine models, and the same will be true of M' and M' will be a model. Thus it suffices to prove the theorem in the case $M = A_i$ is an affine model. Let $\mathfrak{o}$ be the affine ring of M and let F be the complements of M' in M. Then there exists an ideal $\mathfrak{a}$ of $\mathfrak{o}$ which defines F by Theorem 7. Let $a_1, \cdots, a_r$ be nonzero elements of $\mathfrak{o}$ which generate $\mathfrak{a}$ (of $\mathfrak{a} = 0$, then $F = M$ and M' is empty, which is not the case). Let A'_j be the affine model defined by $\mathfrak{o}'_j = \mathfrak{o}[a_j^{-1}]$ for each j and set $M'' = \bigcup_j A'_j$. Each set A'_j being contained in M, M'' is contained in M and M'' is a model. Therefore it is sufficient to show that M' coincides with M''. Let $\mathfrak{p}$ be a prime ideal of $\mathfrak{o}$. Then $\mathfrak{o}_{\mathfrak{p}} \varepsilon M'$ if and

only if $\mathfrak{p}$ does not contain $\mathfrak{a}$. Equivalently, there exists one i such that $a_i \not\in \mathfrak{p}$, i.e., $a_i^{-1} \varepsilon \mathfrak{o}_\mathfrak{p}$ and $\mathfrak{o}_\mathfrak{p} \varepsilon A'_i$, which is equivalent to say that $\mathfrak{o}_\mathfrak{p} \varepsilon M''$. Thus $M' = M''$ and the assertion is proved.

LEMMA 1. *Let A and A' be affine models of the same function field L. Assume that A dominates A'. Then $A \cap A'$ is a model.*

Proof. Let $\mathfrak{o}$ and $\mathfrak{o}'$ be the affine rings of A and A' respectively. Since A dominates A', $\mathfrak{o}$ contains $\mathfrak{o}'$. Take elements $a_1, \cdot \cdot \cdot, a_n$ of $\mathfrak{o}$ such that $\mathfrak{o} = \mathfrak{o}'[a_1, \cdot \cdot \cdot, a_n]$. We set $\mathfrak{a} = \{a ; a \varepsilon \mathfrak{o}', aa_i \varepsilon \mathfrak{o}' \text{ for every } i\}$. Then $\mathfrak{a}$ is an ideal of $\mathfrak{o}'$. Let F be the closed set of A' defined by $\mathfrak{a}$. It is sufficient to show that F coincides with the complements F' of $A \cap A'$ in A' by virtue of Theorem 9 (observe that $A \cap A'$ is not empty because L is in every model of L). Let $\mathfrak{p}'$ be a prime ideal of $\mathfrak{o}'$. Then $\mathfrak{o}'_{\mathfrak{p}'} \varepsilon F$ if and only if $\mathfrak{a} \subseteq \mathfrak{p}'$. Equivalently, there exists one i such that $a_i \not\in \mathfrak{o}'_{\mathfrak{p}'}$ and $\mathfrak{o}'_{\mathfrak{p}'} \varepsilon F'$. Thus $F = F'$ and the lemma is proved.

THEOREM 10. *Let M and M' be models of the same function field L. Then $M \cap M'$ is also a model of L.*

Proof. Since $M \cap M' = J(M, M') \cap M'$, we may assume without loss of generality that M dominates M'. Let A_{ij} and A'_i $(i = 1, \cdot \cdot \cdot, m; j = 1, \cdot \cdot \cdot, n(i))$ be affine models of L such that $M = \bigcup_{ij} A_{ij}$, $M' = \bigcup_i A'_i$ and $A_{ij} \geqq A'_i$ for every i and j (Lemma 2.4.2). Then by the preceding lemma, we see that $A_{ij} \cap A'_i$ is a model, which shows that $M'' = \bigcup_{ij} (A_{ij} \cap A'_i)$ is a model (because it is a subset of a model and is the union of a finite number of affine models). Therefore it is sufficient to show that M'' coincides with $M \cap M'$. Assume that $P \varepsilon M \cap M'$. Then P is in some A_{ij}. Since A_{ij} dominates A'_i, A'_i contains a spot P' which is dominated by P. Since P is in M', we have $P' = P$ and $P \varepsilon A_{ij} \cap A'_i \subseteq M''$. Thus $M \cap M' \subseteq M''$. Since the converse inclusion is obvious, we have $M \cap M' = M''$ and $M \cap M'$ is a model.

COROLLARY 1. *Let M be a model. Then the set M' of normal spots in M is a model.*

The proof is easy if we observe that $M' = M \cap N(M)$.

COROLLARY 2. *Let M and M' be models of function fields L and L' respectively. Assume that L contains L'. Then the set M'' of spots in M which dominate spots in M' is a model.*

The proof is easy if we observe that $M'' = M \cap J(M, M')$.

Now we shall add some remarks on closed sets.

1) If F is a closed set of a model M, F is the union of a finite number of closed sets $F_1, \cdots, F_n$ of M which are irreducible. Here, if there is no inclusion relation $F_1, \cdots, F_n$, we say that each F_i is an *irreducible component* of F. The set of irreducible components of a closed set is uniquely determined. (The proof is similar to that of the remark at the end of § 6.)

2) If E is an irreducible set in a model M and if P is the generating spot of E, then the closure of E in M is the locus of P in M, which shows that it is an irreducible set. (The proof is easy.)

3) With the same notations as in the remark at the end of § 6, the closure of E is the union of the closures of $E_1, \cdots, E_n$. Further, each of the closure of $E_1, \cdots, E_n$ is an irreducible component of the closure of E. (The proof is easy by virtue of the result in 2) above.)

4) By virtue of Proposition 6, we see that the minimum condition for closed sets holds in a model. Consequently, a model is a compact space.

8. Induced models, reduced models and local models. Let P be a quasi-local ring with maximal ideal $\mathfrak{m}$. Then the natural homomorphism from P onto $P/\mathfrak{m}$ is called the *homomorphism defined by* P; it will be denoted by ϕ_P. If M is a set of rings contained in P, the set of homomorphic images of rings in M under ϕ_P (which will be called merely the image of M by ϕ_P) will be denoted by $\phi_P(M)$.

Let M be a model of a function field L over a ground ring I.

PROPOSITION 7. *Let F be an irreducible closed set of M with generating spot P. Then $\phi_P(F)$ is a model of $\phi_P(P)$ over $\phi_P(I)$.*

This model $\phi_P(F)$ is called the *induced model* defined by F, or the induced model of M defined by P; this may be denoted also by $\phi_P(M)$ (because this is uniquely determined by M and P).

Proof. If M is the affine model defined by an affine ring $\mathfrak{o}$, then $\phi_P(F)$ is the affine model defined by $\phi_P(\mathfrak{o})$ over $\phi_P(I)$. From this, in the general case, it follows that $\phi_P(F)$ is the union of a finite number of affine models over $\phi_P(I)$. Assume that two spots $\phi_P(Q)$ and $\phi_P(R)$ in $\phi_P(F)$ $(Q, R \varepsilon F)$ correspond to each other. Then there exists a place $\mathfrak{v}'$ of $\phi_P(P)$ over $\phi_P(I)$ which dominates both $\phi_P(Q)$ and $\phi_P(R)$. Let $\mathfrak{v}$ be a place of L which dominates P and let $\mathfrak{o}$ be a valuation ring of the residue class field of $\mathfrak{v}$ which dominates $\mathfrak{v}'$. Then the composite $\mathfrak{v}''$ of $\mathfrak{v}$ and $\mathfrak{o}$ dominates both Q and R;

Q corresponds to R, whence $Q = R$ and $\phi_P(Q) = \phi_P(R)$. This completes our proof.

Remark. The above proof shows also that spots in F are mapped in a one-to-one way onto spots in $\phi_P(F)$.

Let M be a model over a ground ring I and let $\mathfrak{p}$ be a prime ideal of I. A spot P in M is called a general spot over $\mathfrak{p}$ (or over the ground place $I_{\mathfrak{p}}$) if 1) P dominates $I_{\mathfrak{p}}$ and 2) P is of rank 0 or 1 according as $\mathfrak{p} = 0$ or not. (If $\mathfrak{p} = 0$, then P is just the function field of M.) If, for a given prime ideal $\mathfrak{p}$ of I, there exists one and only one general spot P over $\mathfrak{p}$, then the induced model $\phi_P(M)$ is called the *induced model of M modulo $\mathfrak{p}$*.

Let I' be a subring of L such that 1) there exists an affine ring $\mathfrak{S}$ over I such that I' is a ring of quotients of $\mathfrak{S}$ and 2) I' is a Dedekind domain. Since $\mathfrak{S}$ satisfies the finiteness condition for integral extensions by Theorem 1.3, I' also does and I' is a ground ring.

PROPOSITION 8. *Let $M_{I'}$ be the set of spots in M which contain I'. Then $M_{I'}$ is a model over I'.*

This model $M_{I'}$ is called the *reduced model* of M over I'.

Proof. Since $M_{I'}$ is a subset of a model (over I), we have only to show that $M_{I'}$ is the union of a finite number of affine models over I'. Therefore we may assume that M is an affine model. On the other hand, let A be the affine model defined by the affine ring $\mathfrak{S}$. Then $M_{I'}$ is contained in $J(A, M)$ and $M_{I'} = [J(A, M) \cap M]_{I'}$. Therefore we may assume further that the affine ring $\mathfrak{o}$ of M contains $\mathfrak{S}$. Let M' be the affine model defined by $I'[\mathfrak{o}]$ over I'. Since $I'[\mathfrak{o}]$ is a ring of quotients of $\mathfrak{o}$, M' is a subset of $M_{I'}$. Conversely, if a spot P is in $M_{I'}$, then P is a ring of quotients of $I'[\mathfrak{o}]$ and is in M'. Thus $M_{I'} = M'$ and the assertion is proved.

Now let $\mathfrak{p}$ be a prime ideal of I. Then the reduced model of M over $I_{\mathfrak{p}}$ is called the *local model* of M attached to the ground place $I_{\mathfrak{p}}$; it will be denoted by $M_{\mathfrak{p}}$.

Remark 1. If I is a field, then $\mathfrak{p}$ must be zero, and $M_{\mathfrak{p}} = M$.

Remark 2. If I is a semi-local ring, then any local model is again a model over I.

For, any ground place of I is an affine ring over I.

Remark 3. If I is a Dedekind domain with infinitely many prime ideals, then for any prime ideal $\mathfrak{p}$ of I, the local model $M_{\mathfrak{p}}$ is not a model over I, but is the intersection of infinitely many models over I.

Proof. Let $\boldsymbol{k}$ be the field of quotients of I. Assume that $M_{\mathfrak{p}}$ is a model over I. Let A' be an affine model over I which is contained in $M_{\mathfrak{p}}$ and let $\mathfrak{o}'$ be the affine ring of A'. Since $\boldsymbol{k}$ is an affine ring over $I_{\mathfrak{p}}$ and since $\mathfrak{o}'$ contains $I_{\mathfrak{p}}$, $\mathfrak{o}=\boldsymbol{k}[\mathfrak{o}']$ is an affine ring over I. Let A be the affine model over I defined by $\mathfrak{o}$. Then every spot in A has dimension at least 1, which is a contradiction by virtue of Proposition 3. Therefore $M_{\mathfrak{p}}$ is not a model over I. Now we prove the last assertion. For every maximal ideal $\mathfrak{q}$ of I other than $\mathfrak{p}$, we take an element a of $\mathfrak{q}$ which is not in $\mathfrak{p}$ and set $I((\mathfrak{q}))=I[1/a]$. Let $M(\mathfrak{q})$ be the reduced model of M over $I((\mathfrak{q}))$. Then each $M(\mathfrak{q})$ is a model over I and $M_{\mathfrak{p}}$ is the intersection of all $M(\mathfrak{q})$, which proves the last assertion.

9. Equivalence of the notions of models and of abstract varieties. Let V be a variety (in the sense of Weil [12]) defined over a field $\boldsymbol{k}$; we shall call such a variety an *affine variety*. Let (x) be a generic point of V over $\boldsymbol{k}$. Then the field $\boldsymbol{k}(x)$ is uniquely determined to within isomorphisms over $\boldsymbol{k}$ and is, by definition, a regular extension of $\boldsymbol{k}$. For a point P of V, we denote by $Q_V(P)$ the specialization ring of P in $\boldsymbol{k}(x)$ over $\boldsymbol{k}$. Then the set $S(V)$ of specialization rings $Q_V(P)$ $(P \varepsilon V)$ is the affine model defined by the affine ring $\boldsymbol{k}[x_1,\cdots,x_n]$ (where $(x)=(x_1,\cdots,x_n)$). Further, it is evident that for two points P and P' of V, P' is a specialization of P over $\boldsymbol{k}$ if and only if $Q_V(P')$ is a specialization of $Q_V(P)$. Conversely, if M is the affine model of a function field L over a ground field $\boldsymbol{k}$ defined by an affine ring $\boldsymbol{k}[x_1,\cdots,x_n]$ and if L is a regular extension of $\boldsymbol{k}$, and if V is the affine variety defined over $\boldsymbol{k}$ with generic point $(x_1,\cdots,x_n)$, then $S(V)=\{Q_V(P);\ P \varepsilon V\}$ coincides with M.

Next, let V and V' be birationally equivalent affine varieties and let T be a birational correspondence between V and V'. Assume that V, V' and T are defined over a field $\boldsymbol{k}$. Let (x) and (x') be corresponding generic points of V and V' under T. We can identify $\boldsymbol{k}(x)$ and $\boldsymbol{k}(x')$ by the correspondence T. Then it will be easily seen that the join of the affie models defined by $\boldsymbol{k}[(x)]$ and $\boldsymbol{k}[(x')]$ corresponds to T in the sense we stated above and that a point P in V corresponds biregularly to a point P' in V' if and only if $Q_V(P)=Q_{V'}(P')$, provided that P and P' are corresponding points under T. (Observe that if $Q_V(P)=Q_{V'}(P')$ for points P and P' in V and V' respectively, then there exists a point P'' in V' such that P'' is a generic specialization of P' over $\boldsymbol{k}$ and that P'' corresponds biregularly to P under T.)

Now let V be an abstract variety (in the sense of Weil [12]) defined over a field $\boldsymbol{k}$, with representatives V_i, frontiers $\mathfrak{F}_i$ and birational corre-

spondences T_{ij}. Let $(x^{(i)})$ be generic points of the varieties V_i which correspond to each other under the correspondences T_{ij}. Then we can identify the fields $\boldsymbol{k}(x^{(i)})$ with each other. Let L be the field obtained in this manner. Now let $P^{(i)}{}_1, \cdots, P^{(i)}{}_{n(i)}$ be points in $\mathfrak{F}_i$ such that every point of $\mathfrak{F}_i$ is a specialization of one of $P^{(i)}{}_1, \cdots, P^{(i)}{}_{n(i)}$ over $\boldsymbol{k}$; such points exist because $\mathfrak{F}_i$ is normally algebraic over $\boldsymbol{k}$ by definition. Then $S(\mathfrak{F}_i) = \{Q_{V_i}(P); P \varepsilon \mathfrak{F}_i\}$ is the union of the loci of the spots $Q_{V_i}(P^{(i)}{}_j)$ in the model $S(V_i)$ and is a closed set of $S(V_i)$. Therefore, $S(V_i - \mathfrak{F}_i) = \{Q_{V_i}(P); P \varepsilon V_i - \mathfrak{F}_i\}$ is a model of L by Theorem 9. Let P be a point of V. Then there exists one i such that P has a representative P_i in $V_i - \mathfrak{F}_i$. We call the specialization ring $Q_{V_i}(P_i)$ of P_i the specialization ring of P and denote it by $Q_V(P)$; even if P has another representative P_j in $V_j - \mathfrak{F}_j$, $Q_V(P)$ is unique because P_i and P_j correspond biregularly to each other. Now let $S(V)$ be the set of specialization rings of points in V. Then $S(V)$ is the union of all $S(V_i - \mathfrak{F}_i)$. Since $S(V_i - \mathfrak{F}_i)$ is a model, it is the union of a finite number of affine models, and the same is true of $S(V)$. To prove that $S(V)$ is a model, we have only to show that any two different spots in $S(V)$ do not correspond to each other. Assume that a spot $Q_V(P)$ corresponds to a spot $Q_V(P')$ $(P, P' \varepsilon V)$. Let P_i and P'_j be representatives of P and P' in $V_i - \mathfrak{F}_i$ and $V_j - \mathfrak{F}_j$ respectively. Then there exists a point P'' in $V_j - \mathfrak{F}_j$ which corresponds to P_i under T_{ij} such that P'' is a generic specialization of P'_j. Since P'' corresponds to P_i under T_{ij}, P'' is a representative of P, whence $Q_V(P) = Q_{V_j}(P'') \varepsilon S(V_j - \mathfrak{F}_j)$. Since $S(V_j - \mathfrak{F}_j)$ is a model, we have $Q_V(P) = Q_V(P')$ and $S(V)$ is a model.

Conversely, assume that M is a model of a function field L over a field $\boldsymbol{k}$ and assume that L is a regular extension of $\boldsymbol{k}$. Let $A_1, \cdots, A_n$ be affine models of L such that M is the union of them. Let $\mathfrak{o}_i = \boldsymbol{k}[x^{(i)}{}_1, \cdots, x^{(i)}{}_{m(i)}]$ be the affine ring of A_i. Let V_i be the affine variety of generic point $(x^{(i)}) = (x^{(i)}{}_1, \cdots, x^{(i)}{}_{m(i)})$ defined over $\boldsymbol{k}$ and let T_{ij} be the birational correspondence between V_i and V_j under which $(x^{(i)})$ and $(x^{(j)})$ correspond to each other. Then there exists an abstract variety V with representatives V_i, birational correspondences T_{ij} and empty frontiers. The variety V being defined in this manner, we see easily that $S(V)$, as defined above, coincides with M.

Remark. Let V be an abstract variety defined over a field $\boldsymbol{k}$. Then there exists an abstract variety V' which is everywhere in biregular correspondence with V and which has representatives V_i and correspondences T_{ij} such that the frontiers are empty.

Appendix. A proof of a result due to Krull.

LEMMA 1. *Let* $\mathfrak{o}$ *be a complete local integral domain of rank* r. *If* $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_s$ *is a maximal chain of prime ideals in* $\mathfrak{o}$ *(that is,* $\mathfrak{p}_0 = 0$, $\mathfrak{p}_s$ *is maximal and each* $\mathfrak{p}_i/\mathfrak{p}_{i-1}$ *is of rank* 1 *for every* $i = 1, \cdots, s$*), then* $r = s$. (Cohen [2])

Proof. Let $\mathfrak{r}$ be an unramified regular local ring contained in $\mathfrak{o}$ such that $\mathfrak{o}$ is a finite $\mathfrak{r}$-module (Lemma 0.8). Since a regular local ring is a normal ring (Lemma 0.7) and since $\mathfrak{o}$ is an itegral extension of $\mathfrak{r}$, $\mathfrak{p}_1 \cap \mathfrak{r}$ is of rank 1 ([10, § 5]). Therefore $\mathfrak{p}_1 \cap \mathfrak{r}$ is a principal ideal (Lemma 0.9),[4] whence rank $(\mathfrak{r}/\mathfrak{p}_1 \cap \mathfrak{r})) = r - 1$ and rank $\mathfrak{o}/\mathfrak{p}_1 = r - 1$. Thus we prove our assertion easily by induction on r.

Now we prove

LEMMA 2. *Let* $\mathfrak{r}$ *be a regular local ring of rank* n. *If* $\mathfrak{p}$ *is a prime ideal of* $\mathfrak{r}$, *then* rank $\mathfrak{p}$ + co-rank $\mathfrak{p} = n$. (Krull [5])

Proof. Let $\mathfrak{r}^*$ be the completion of $\mathfrak{r}$. Then $\mathfrak{r}^*$ is also a regular local ring, and it is an integral domain (Lemma 0.7). Set rank $\mathfrak{p} = r$, co-rank $\mathfrak{p} = s$. Since $\mathfrak{r}^*/\mathfrak{p}\mathfrak{r}^*$ is the completion of $\mathfrak{r}/\mathfrak{p}$, rank $\mathfrak{r}^*/\mathfrak{p}\mathfrak{r}^* = s$ (Lemma 0.1), which shows that there exists a prime divisor $\mathfrak{p}^*$ of $\mathfrak{p}\mathfrak{r}^*$ such that co-rank $\mathfrak{p}^* = s$ and that there exists no prime divisor of $\mathfrak{p}\mathfrak{r}^*$ whose co-rank is greater than s. Therefore rank $\mathfrak{p}\mathfrak{r}^* = n - s$ by Lemma 1. On the other hand, let S be the complements of $\mathfrak{p}$ in $\mathfrak{r}$. Since $\mathfrak{r}^*/\mathfrak{p}\mathfrak{r}^*$ is the completion of $\mathfrak{r}/\mathfrak{p}$, every element of $\mathfrak{r}/\mathfrak{p}$ is not a zero-divisor in $\mathfrak{r}^*/\mathfrak{p}\mathfrak{r}^*$ (Corollary to Lemma 0.4), which shows that every element of S is not in any prime divisor of $\mathfrak{p}\mathfrak{r}^*$. Therefore rank $\mathfrak{p}\mathfrak{r}^* =$ rank $\mathfrak{p}\mathfrak{r}^*_S$. Let $a_1, \cdots, a_r$ be a system of parameters of $\mathfrak{r}_S = \mathfrak{r}_{\mathfrak{p}}$. Then $\mathfrak{p}\mathfrak{r}^*_S$ and $\sum a_i\mathfrak{r}^*_S$ have the same radical, whence rank $\mathfrak{p}\mathfrak{r}^*_S =$ rank $\sum a_i\mathfrak{r}^*_S$. Therefore rank $\mathfrak{p}\mathfrak{r}^*_S \leqq r$; it follows that rank $\mathfrak{p}\mathfrak{r}^* \leqq r$. Therefore $n - s \leqq r$. Since obviously $n \geqq r + s$, we see that $n - s = r$ and the assertion is proved.

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[4] If we want to avoid making use of Lemma 0.9, we may choose $\mathfrak{r}$ such that some element x of $\mathfrak{p}_1$ is in $\mathfrak{r}$ but not in the square of the maximal ideal of $\mathfrak{r}$. Then $\mathfrak{p}_1 \cap \mathfrak{r}$ is generated by x.

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ON THE CURVATURES OF A SURFACE.*

By Aurel Wintner.

I. On the Mean Curvature.

1. Let S be a (piece of a) surface in the X-space, where $X=(x,y,z)$, and let $S \varepsilon C^n$ for a fixed $n>0$. By this is meant that a neighborhood of every point of S has *some* C^n-paramerization, that is, a parametrization of the form

$$S\colon\ X=X(u,v), \qquad (u,v)\,\varepsilon\, D, \tag{1}$$

where D is an open, simply connected (u,v)-domain and $X(u,v)$ a vector function satisfying the conditions $X(u,v)\,\varepsilon\, C^n$ and $[X_u, X_v]\neq 0$ on D. Note that if $n<m$, then a C^n-parametrization of an $S\,\varepsilon\, C^m$ need not be a C^m-parametrization. If $S\,\varepsilon\, C^1$, there exists on S a continuous normal vector N. The latter is a function $N(u,v)\,\varepsilon\, C^{n-1}$ in terms of every C^k-parametrization (1) of an $S\,\varepsilon\, C^n$ not only if $k=n$ but also if $k=n-1$ (but not necessarily if $k=n-2$). In order that $S\,\varepsilon\, C^n$, where $n>1$, it is sufficient (and, of course, necessary) that S possesses some parametrization (1) satisfying

$$X(u,v)\,\varepsilon\, C^{n-1} \text{ and } N(u,v)\,\varepsilon\, C^{n-1}, \text{ where } [X_u, X_v]\neq 0. \tag{2}$$

In fact, the sufficiency of (2) for *some* C^n-parametrization of S is a consequence of the (local) theorem on implicit functions; cf., e.g., [8], pp. 133-134, and the references given there.

If $\alpha=\alpha(u,v)$ denotes the matrix of the first, and $\beta=\beta(u,v)$ that of the second, fundamental matrix in terms of a C^2-parametrization (1) of an $S\,\varepsilon\, C^2$, then α is a (symmetric) positive definite matrix satisfying $\alpha(u,v)\,\varepsilon\, C^1$ but nothing more than $\beta(u,v)\,\varepsilon\, C^0$ is in general true of the (symmetric) matrix β; so that the mean curvature $H=H(u,v)$ exists but is, in general, just continuous, since it is defined by

$$H=\tfrac{1}{2}\operatorname{tr}(\beta\alpha^{-1}); \tag{3}$$

in view of

$$K=\det(\beta\alpha^{-1}), \tag{4}$$

* Received July 27, 1955.

this applies to the Gaussian curvature $K = K(u, v)$ also. As is well-known,

$$K \leqq H^2, \tag{5}$$

where the sign of equality holds at a point (u, v) if and only if (u, v) is an umbilical point, that is, a point at which β becomes a scalar multiple of α (a "flat" point, with $K = 0 = H$, is here included as "umbilical").

The following considerations on H, considerations in which $S \varepsilon C^2$ or, if $S \varepsilon C^2$, the C^2-character of a given parametrization (1) of $S \varepsilon C^2$ will not be assumed, have various motivations. On the one hand, it can happen that no C^2-parametrization of a given $S \varepsilon C^2$ is available, since precisely the "natural" parametrization of $S \varepsilon C^2$, one given in geometrical terms, must fail to be a C^2-parametrization (cf. Section 8 below); so that the definition of (3) of H fails (in terms of what is available). On the other hand, the theory of minimum surface ($H \equiv 0$) speaks since Haar (cf. [5]) of surfaces $S \varepsilon C^1$ for which $S \varepsilon C^2$ is not assumed (even though proved, as a consequence of the theory). Such occurrences call for a definition of $H = H(u, v)$ which is more geometrical than (3) (concerning the geometrical, but erroneous, attempts of Minding and R. Sturm, cf. the comments of Stäckel [15]).

An approach of the desired type can be abstracted from the manner in which H appears when, in the classical instance of varying a double integral, the first variation,

$$\delta \iint_B |[X_u, X_v]| \, du dv, \tag{6}$$

of the area (of a piece of S) is transformed into a line integral.

2. Let (1) be a C^1-parametrization of an $S \varepsilon C^1$; so that $N(u, v) \varepsilon C^0$ holds for the unit vector

$$N = [X_u, X_v]/g, \text{ where } g = |[X_u, X_v]| > 0 \tag{7}$$

(this g is the positive square root of the determinant of

$$|dX(u, v)|^2 = E(u, v) du^2 + 2F(u, v) du dv + G(u, v) dv^2, \tag{8}$$

the first fundamental form of S, with continuous E, F, G). By the existence of a continuous mean curvature on $S \varepsilon C^1$ should be meant that, for *some* continuous vector function $(\cdot \cdot \cdot)$ of (u, v),

$$\int_J [N, X_u du + X_v dv] = \iint_B (\cdot \cdot \cdot) du dv \tag{9}$$

holds as an identity in J and its interior $B = B(J)$. Here J denotes any positively oriented, piecewise smooth Jordan curve which, together with B, is contained in the fixed (u, v)-domain D on which the C^1-function (1) is given.

It is easy to realize that the existence of a continuous $(\cdot\ \cdot\ \cdot)$ satisfying the requirement (9) is not implied by the C^1-character of (1). If there exists a continuous $(\cdot\ \cdot\ \cdot)$, it is unique, since B is arbitrary. It also follows that $(\cdot\ \cdot\ \cdot)$ must be a vector function which is a (possibly vanishing) scalar multiple of $N(u, v)$, simply because the scalar product of $dX = X_u du + X_v du$ and N is 0; cf. (7). Hence $(\cdot\ \cdot\ \cdot)$ can be written as gN times $-2H$, where $H = H(u, v)$ is a unique continuous scalar, since $g = g(u, v)$ in (7) is positive and continuous. Let H be declared to be the mean curvature of S. The explicit form of (9) becomes

$$\int_J [N, dx] = \iint_B -2HgN\,du\,dv. \tag{10}$$

3. This transcribes into a (generalized) definition a fact pointed out by Weatherburn ([16], p. 255; cf. also [14]), who observed that (10) holds on every $S \varepsilon C^2$ if H is defined by (3). For reasons of covariance, (10) holds for C^1-parametrizations (1) of $S \varepsilon C^2$ also. Actually, it is clear from (7) that, for reasons of covariance, (10) holds in every C^1-parametrization of every $S \varepsilon C^1$ if it holds in one C^1-parametrization of that $S \varepsilon C^1$, and that H is invariant if a C^1-parametrization (1) is replaced by any other C^1-parametrization of $S \varepsilon C^1$ (provided that the local C^1-transformation, say

$$u^* = u^*(u, v), \qquad v^* = v^*(u, v), \tag{11}$$

which represents this reparametrization is of positive Jacobian; if the Jacobian of (11) is negative, then N goes over into $-N$, hence H into $-H$, and so only H^2 is invariant).

If $S \varepsilon C^2$, and if (1) is a C^2-parametrization of S, then Gauss' definition of K (in terms of oriented areas of the spherical images of portions of S), when combined with the derivation formulae of Weingarten (cf., e.g., [2], p. 62), can be written in the form

$$\int_J [N, dN] = \iint_B -2KgN\,du\,dv. \tag{12}$$

This striking analogue of (10), also pointed out in Weatherburn's paper [16], can be considered as defining K in the same way as (10) defines H. But

(12), in contrast to (10), differentiates $N(u,v)$ and represents, therefore, a requirement which cannot be formulated if only $X(u,v)\,\varepsilon\, C^1$ is assumed. On the other hand, (12) holds whenever (1) is a C^1-parametrization of an $S\,\varepsilon\, C^2$. This follows for reasons of invariance, and since $N(u,v)\,\varepsilon\, C^1$ holds for every C^1-parametrization (1) of an $S\,\varepsilon\, C^2$ (conversely, the case $n=2$ of the assumptions (2) implies that $S\,\varepsilon\, C^n = C^2$).

A definition of the mean curvature which is more general than the definition of a *continuous* H (on an $S\,\varepsilon\, C^1$) above results as follows: With reference to any piecewise smooth J, put

$$\phi(B) = \int_J [N, dX], \tag{13}$$

where B is the interior of J. Suppose that the set-function (13) can be extended from the particular (u,v)-sets $B = B(J)$ to all Borel sets E, leading to an additive set-function $\phi(E)$, defined on the Borel field of (u,v)-sets E contained in D. Then, if $\phi(E)$ is absolutely continuous, the analogue of the relation (10) leads to an L-integrable $H(u,v)$.

4. It will now be proved that an $S\,\varepsilon\, C^1$ possessing a continuous H need not be an $S\,\varepsilon\, C^2$; in other words, that the definition, (10), of Section 2 is actually more general than the classical definition, (3). In fact, the example $S\,\varepsilon\, C^1$ to be given will be such that $\beta(u,v)$ will in no sense exist at a point, say $(u,v) = (0,0)$, although there will exist a continuous H. Incidentally, this particular S will be such as to possess a C^1-parametrization in which $\alpha(u,v)\,\varepsilon\, C^1$ (as though (1) were a C^2-parametrization, which (1) cannot be, since $S\,\varepsilon\, C^2$ fails to hold).

First, if an $S\,\varepsilon\, C^2$ is given in a Cartesian parametrization, say as $z = z(x,y)$ (so that $u = x$, $v = y$), then $z(x,y)\,\varepsilon\, C^2$ on an (x,y)-domain, D, and (3) reduces to

$$(1+q^2)r - 2pqs + (1+p^2)t = 2Hh^3 \tag{14}$$

where $p = z_x, \cdots, t = z_{yy}$ and

$$h = (1+p^2+q^2)^{\frac{1}{2}}. \tag{15}$$

Next, if $[F]$ denotes the Lagrangian derivative of an $F = F(x,y,z,p,q)$, then, formally, the identical vanishing of (6) is equivalent to the case $F = h$ of $[F] = 0$, whereas Laplace's equation, $r + t = 0$, belongs to $F = p^2 + q^2$. Since $[F+G] = [F] + [G]$, and since $[G] = -f(x,y)$ when $G =$

$G(x, y, z, p, q)$ is of the form $G = zf(x, y)$, the formal analogy between (14) and Poisson's equation

$$(16) \qquad r + t = f(x, y)$$

indicates that, in order to obtain an $S \varepsilon C^1$ of the desired type, a construction of Lichtenstein [10], which refers to $F = p^2 + q^2 - 2zf(x, y)$, can be adapted.

Lichtenstein's construction is based on the results of Petrini [12] on logarithmic potentials. The following construction will have the same basis, in a form similar to that used in [8], pp. 134-135.

5. Let D be the unit circle $0 \leqq x^2 + y^2 < 1$, and let $w(x, y)$ be $(-2\pi)^{-1}$ times the logarithmic potential of a density $f(x, y)$ which is uniformly continuous on D. According to Petrini, $w(x, y) = w_f(x, y)$ need not possess second derivatives $r, \cdots$ (so that (16) need not be meaningful for $z = w$), if f is suitably chosen on D, an example being

$$(17) \qquad f(x, y) = (\cos\theta)^2/\log\rho \qquad (x = \rho\cos\theta, y = \rho\sin\theta)$$

(if $0 < \rho$, with $f(0, 0) = 0$ at $\rho = 0$); cf. [12], p. 138. But $z = w$ must always satisfy the "integrated form" of (16), which is

$$(18) \qquad \int_J (pdy - qdx) = \iint_B fdxdy$$

(cf. [10], pp. 99-100), where J and $B = B(J)$ are meant in the same sense as in (9). It is also known that, since $f(x, y)$ is uniformly continuous, $p = w_x$ and $q = w_y$ will not only exist but be such as to satisfy a uniform Hölder condition of any index $\lambda < 1$; in particular, of some index $\lambda > \frac{1}{2}$. But the analyticity of (17) for $\rho \neq 0$ implies that $w_f(x, y)$ is analytic, hence of class C^2, at every point $(x, y) \neq (0, 0)$ of D. On the other hand, since $w(x, y)$ is the logarithmic potential of a continuous density, it is readily seen that $w_x(x, y) = O(|\log r|)$ and $w_y(x, y) = O(|\log r|)$ as $r = (x^2 + y^2)^{\frac{1}{2}} \to 0$ (in this connection, cf. [18], pp. 736-737). Hence, if $p = w_x$ and $q = w_y$ are such as to satisfy

$$(19) \qquad p(0, 0) = 0 \text{ and } q(0, 0) = 0,$$

then

$$(20) \qquad p^2(x, y) \varepsilon C^1, \quad p(x, y)q(x, y) \varepsilon C^1, \quad q^2(x, y) \varepsilon C^1 \text{ on } D\colon 0 \leqq x^2 + y^2 < 1$$

(even though $p(x, y) \varepsilon C^1$, $q(x, y) \varepsilon C^1$ cannot hold, since $w(x, y) \varepsilon C^2$ is false). Finally, (19) becomes satisfied, and both (20) and the formulation (18) of

(16) remain valid, if $w(x,y)$ in the preceding deduction is replaced by $z(x,y)+cx+dy$, where $c=-w_x(0,0)$, $d=-w_y(0,0)$.

Accordingly, S: $z=z(x,y)$ is of class C^1 but not of class C^2 (even though the coefficients of (8), where $(u,v)=(x,y)$, are functions of class C^1, since

$$(21) \qquad E=1+p^2, \qquad F=pq, \qquad G=1+q^2,$$

and since (20) holds). It remains to be shown that this S has a continuous $H=H(x,y)$.

6. A Pfaffian

$$(22) \qquad a(x,y)\,dx+b(x,y)\,dy$$

is called regular (on D) if $a(x,y)$, $b(x,y)$ are continuous and such as to satisfy the condition

$$(23) \qquad \int_J (adx+bdy)=\iint_B fdxdy$$

for *some* continuous $f(x,y)$, where $J, B=B(J)$ have the same meaning as in (9). It is clear from Green's theorem that (22) must be regular if $a(x,y)\,\varepsilon\,C^1$ and $b(x,y)\,\varepsilon\,C^1$. It is also known that if (22) is regular (for some reason), then $\mu(x,y)$ times the Pfaffian (22) must also be regular whenever

$$(24) \qquad \mu(x,y)\,\varepsilon\,C^1,$$

since (23) and (24) imply that

$$(25) \qquad \int_J \mu\cdot(adx+bdy)=\iint_B (\mu f-a\mu_y+b\mu_x)\,dxdy.$$

This is a theorem of E. Cartan (which unfortunately escaped us in [6], where, without a reference to [3], pp. 69-70, it is stated (p. 761) and, essentially with E. Cartan's proof, is proved as a lemma).

Let $\mu=1/h$. Then (15) and (20) show that (24) is satisfied. Hence, if (23) is identified with (18), the case $\mu=1/h$ of (25) is applicable and leads to

$$(26) \qquad \int_J h^{-1}\cdot(z_xdy-z_ydx)=\iint_B (\cdot\ \cdot\ \cdot)\,dxdy,$$

where $(\cdot\ \cdot\ \cdot)$ is a certain continuous function of (x,y). On the other hand, since the C^1-parametrization (1), where $X=(x,y,z)$, is now given in the form

$$(27) \qquad S:\ z=z(x,y),$$

it is clear that the vector requirement (10) for a continuous H reduces to the scalar condition

$$\int_J (1 + z_x^2 + z_y^2)^{-\frac{1}{2}}(z_x dy - z_y dx) = \int\int_B 2H dx dy. \tag{28}$$

Since the existence of a continuous H satisfying (28) follows from (26) and (15), this proves the existence of an S which has the properties announced at the beginning of Section 4.

7. A matrix function

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{29}$$

of (u, v) is called regular (on a (u, v)-domain D) if both Pfaffians

$$a(u, v)du + b(u, v)dv, \qquad c(u, v)du + d(u, v)dv$$

are regular. Since this will be the case if (though not only if) the function (29) is of class C^1, the matrix, $\alpha(u, v)$, of the first fundamental form (8) of a C^2-parametrization (1) of an $S \varepsilon C^2$ is a regular matrix. With regard to the matrix, $\beta(u, v)$, of the second fundamental form

$$L(u, v)du^2 + 2M(u, v)dudv + N(u, v)dv^2, \tag{30}$$

the situation is as follows:

The classical formulation of the Mainardi-Codazzi equations is of the form

$$L_v - M_u = (\cdot\ \cdot\ \cdot), \qquad M_v - N_u = (\cdot\ \cdot\ \cdot) \tag{31}$$

and assumes a C^3-parametrization (1) for an $S \varepsilon C^3$. Under this assumption, $\alpha(u, v) \varepsilon C^2$ and $\beta(u, v) \varepsilon C^1$. It follows therefore from Green's identity that (31) is equivalent to the pair of relations

$$\begin{aligned} \int_J (Ldu + Mdv) &= \int\int_B (\cdot\ \cdot\ \cdot)dudv, \\ \int_J (Mdu + Ndv) &= \int\int_B (\cdot\ \cdot\ \cdot)dudv \end{aligned} \tag{32}$$

(as identities in J and its interior B), where the expressions $(\cdot\ \cdot\ \cdot)$ are the same as the respective expressions $(\cdot\ \cdot\ \cdot)$ in (31). Both of these functions $(\cdot\ \cdot\ \cdot)$ of (u, v) are bilinear forms in

$$(L, M, N) \text{ and } (E_u, E_v, F_u, F_v, G_u, G_v), \tag{33}$$

with coefficients which are rational functions of the coefficients of (8). There is no point in writing here down this classical pair of expressions $(\cdot\ \cdot\ \cdot)$, the more so as, in the more general formulation of (32) which will be given under (II) below, $(\cdot\ \cdot\ \cdot)$ *could not be written down* (in fact, the second of the two sets (33) will then become undefined). The preceding assumption, $S\,\varepsilon\,C^3$, is relaxed to $S\,\varepsilon\,C^2$ in (I) and (II) below.

(I) Let (1) be a C^2-parametrization of an $S\,\varepsilon\,C^2$. Then all $3+6$ functions (33) exist and are continuous, since $\beta(u,v)\,\varepsilon\,C^0$, $a(u,v)\,\varepsilon\,C^1$. Moreover, if the two classical expressions $(\cdot\ \cdot\ \cdot)$ are formed from the functions (33) in the same way as in the traditional case $S\,\varepsilon\,C^3$, it can be shown (cf. [6], pp. 760-766) that the Mainardi-Codazzi equations are valid in the form (32). Consequently, $\beta(u,v)$ (and not only $a(u,v)$; cf. above) is a regular matrix if (1) is a C^2-parametrization of an $S\,\varepsilon\,C^2$.

(II) It follows that (32) holds in terms of every C^1-parametrization (1) of an $S\,\varepsilon\,C^2$, provided that the (continuous) elements L, M, N of the (symmetric) matrix function β are defined, not directly (which, in terms of the C^1-parametrization of $S\,\varepsilon\,C^2$, would not be possible), but by covariance, and that the validity of the Mainardi-Codazzi equations is interpreted to mean the existence of *certain* continuous functions $(\cdot\ \cdot\ \cdot)$ for which the pair of relations (32) holds as an identity in J and its interior B. In terms of a C^2-parametrization (1) of $S\,\varepsilon\,C^2$, the second of the sets (33) enters into $(\cdot\ \cdot\ \cdot)$ in the form of the Christoffel coefficients $\Gamma^i{}_{jk}$ of $a(u,v)\,\varepsilon\,C^1$. But $\Gamma^i{}_{jk}$ is not a tensor; its transformation rule involves the *second* derivatives of a C^1-transformation (11) of non-vanishing Jacobian, and these second derivatives will not exist if (11) represents the passage from a C^2-parametrization (1) to a C^1-parametrization $X=X(u^*;v^*)$ of $S\,\varepsilon\,C^2$. On the other hand, the coefficients of (30) form a contravariant tensor under C^1-transformation (11) (of positive Jacobian) and are, therefore, defined in terms of C^1-parametrization of $S\,\varepsilon\,C^2$ also. Since the regularity of any matrix (29) is preserved under C^1-transformations (11) of non-vanishing Jacobian (when (29) is thought of as a tensor which is contravaniant in both indices), the proof is complete.

8. It seems to be quite artificial to consider, as in (II) or (12) and (10), parametrizations (1) of an $S\,\varepsilon\,C^2$ which are just C^1-parametrizations. Actually, such a step is often unavoidable, since it can be imposed by the *geometrical* structure of a problem. A convincing instance of this situation presents itself in the theory of ruled surfaces, a theory which (as it turns out,

partly for this reason) is in a notoriously poor shape from the point of view of analysis.

Let

(34) a ruled surface R of class C^2

be defined as follows: R is an $S \varepsilon C^2$ possessing a C^1-parametrization (1) in which the parameter lines $u = \text{const.}$ correspond to segments $\Lambda = \Lambda(u)$ of straight lines in the X-space, where $X = (x, y, z)$. In other words, (34) means that R is a surface S which, besides possessing *some* C^2-parametrization (1), has a C^1-parametrization of the form

$$(35) \qquad R\colon\ X(u,v) = A(u)v + B(u),$$

where $A(u)$, $B(u)$ are vector functions possessing continuous first derivatives $A'(u)$, $B'(u)$ (on a certain interval $u_0 < u < u^0$).

It would be out of place to replace this definition of (34) by the requirement that $R \varepsilon C^2$ should have a C^2-parametrization of the form (35). For suppose that an R possessing a C^2-parametrization happens to be a "developable" (a "torse"), in the sense that the plane tangent to R at (u,v) does not vary along $\Lambda(u)$. Then it is readily seen from (35), where $A(u) \varepsilon C^2$ and $B(u) \varepsilon C^2$ by assumption, that the normal vector (7) is a function $N(u)$ of class C^2. Since the case $n = 3$ of (2) implies that $S \varepsilon C^3$, it follows that R, instead of being just an $S \varepsilon C^2$, must be an $S \varepsilon C^3$. For a certain converse (even with C^2, C^3 replaced by C^1, C^2, respectively), cf. [8], pp. 133-134. In what follows, neither $N(u,v) = N(u)$ nor $R \varepsilon C^3$, but only (34), will be assumed.

Since (35) is just a C^1-parametrization, the coefficients of the first fundamental form (8) follow from (35) as continuous functions, whereas the second fundamental form cannot be formed in this parametrization. But the (continuous) second derivatives $X_{vv}, X_{uv} = X_{uv}$ of (35) exist (though X_{uu} need not), and so it is easy to realize that the last two coefficients, M and N, of (30) can be defined not only by covariance (as in (II), Section 7), but also by their standard definition. According to the latter, M, N are the scalar products of $[X_u, X_v]$ and X_{uv}/g, X_{vv}/g, respectively, where $g = (EG - F^2)^{\frac{1}{2}} > 0$ (cf., e.g., [2], p. 52). In view of (35), this leads to

$$(36) \qquad M = \det(A, A', B')/g, \qquad N = 0,$$

where $g = |[X_u, X_v]|$ and $' = d/du$. Hence, in order to obtain (30) in terms of the C^1-parametrization (35) of $R \varepsilon C^2$, only $L = L(u,v)$ remains to be determined. But it turns out that L is given by

$$(37) \qquad L = 2g^2H/G + 2F\det(A, A', B')/(gG),$$

where the coefficients of (8) are supplied by the first derivatives of (35) and $H = H(u, v)$ is the mean curvature. The latter is supplied by (10), since (10) is valid in terms of the C^1-parametrization (35) of $R \varepsilon C^2$ also.

In order to verify (37), note that, since the coefficient matrices, α and β, of (8) and (30) are contravariant tensors, (3) holds in terms of C^1-parametrizations of every $S \varepsilon C^2$. But (37) follows if (36) is inserted in the explicit form of (3).

9. In Section 8, reference was made to a paradoxical situation which originates from the conclusion $(2) \rightarrow (S \varepsilon C^n)$. In what follows, the same conclusion will lead to certain, geometrically quite unexpected, implications dealing with the degree of smoothness concerning both types of classical developables, say $S = V = V(\Gamma)$ and $S = W = W(\Gamma)$, which are the envelopes (V) of the normal planes and the envelopes (W) of the rectifying planes of a space curve Γ.

The results to be obtained can be considered as counterparts of the results obtained, in [19], p. 368, and [20], p. 251, respectively, for the parallel surfaces (Steiner) and the evolute surfaces (Monge) of a surface. In all four cases, the result is to the effect that, subject to a rank condition on the Jacobian matrix involved, the "generated" surfaces are smoother than one would expect from their geometrical definitions.

Let Γ: $X = X(s)$, where $X = (x, y, z)$, be a (sufficiently short piece of a) curve of class C^3 possessing a non-vanishing second derivative $X''(s)$ with respect to the arc length s. Then Γ has a positive curvature $\kappa(s) \varepsilon C^1$ and a torsion $\tau(s) \varepsilon C^0$, and the unit vectors $U_1 = X'$, $U_2 = X''/\kappa$, $U_3 = [U_1, U_2]$ satisfy Frenet's equations

$$(38) \qquad U'_1 = \kappa U_2, \qquad U'_2 = -\kappa U_1 + \tau U_3, \qquad U'_3 = -\tau U_2$$

(so that $U_1(s) \varepsilon C^2$ but $U_i(s) \varepsilon C^1$ if $i = 2$ or $i = 3$). Conversely, if any positive function $\kappa(s) \varepsilon C^1$ and any real-valued function $\tau(s) \varepsilon C^0$ are given on an s-interval, then an application of the existence and uniqueness theorem of linear systems of ordinary differential equations supplies a unique $\Gamma \varepsilon C^3$ satisfying (38).

10. It follows that if only $\Gamma \varepsilon C^3$ (with $\kappa > 0$) is assumed, then the function

$$(39) \qquad X(s, t) = X(s) + U_2(s)/\kappa(s) + tU_3(s),$$

where t is a linear parameter and $X = X(s)$ is the parametrization of Γ in

terms of the arc length, need not have a second derivative $X_{ss}(s,t)$. But it turns out that the locus S: $X=X(s,t)$, which, as is well-known (cf., e.g., [2], pp. 43-44), is the envelope $S=V=V(\Gamma)$ defined in Section 9, is an $S\,\varepsilon\,C^2$, provided that the curve, say $a(s,t)=0$, on which

$$[X_s, X_t]\neq 0 \tag{40}$$

fails to hold is excluded from S. The explicit representation of a is

$$a=a(s,t)=t\tau(s)+\kappa'(s)/\kappa^2(s) \tag{41}$$

(if $a(s,t)=0$ is inserted from (41) into (39), it follows that that singular curve on S is the path, if any, of the centres of the osculating spheres of Γ; cf., e.g., [2], p. 36, and, if $\Gamma\,\varepsilon\,C^n$ for some $n\geqq 4$, the theorem italicized on p. 246 of [20], where $n=4$).

Accordingly, the "natural" parametrization, (39), of $S=V(\Gamma)$, being just a C^1-parametrization, disguises the fact that, under the restriction (40), there must exist some C^2-parametrization $X=X(u;v)$. The proof will be such as to show a corresponding result, $S\,\varepsilon\,C^{n-1}$ (instead of just $S\,\varepsilon\,C^{n-2}$) when $\Gamma\,\varepsilon\,C^n$ in $S=V(\Gamma)$, holds for every $n\geqq 3$.

Suppose that $\Gamma\,\varepsilon\,C^3$ and $(\kappa>0)$. Then (38) is applicable. But if (39) is differentiated with respect to s and t and then use is made of (38), where $U_1(s)=X'(s)$, it is seen that

$$[X_s, X_t]=-aU_1, \tag{42}$$

where a is the scalar defined by (41). It follows from (42) that, on the one hand, $S=V(\Gamma)$ has a (unit) normal vector $N=N(s,t)$ unless $a=a(s,t)$ vanishes and that, on the other hand, $N=\pm U_1$ if $a\neq 0$, that is, if (40) is satisfied (it also follows that $N(s,t)=N(s)$, but this will not be used). Since $U_1(s)\,\varepsilon\,C^2$, it follows that the normal vector N is a function of class C^2, and therefore of class C^1, in a C^1-parametrization, (39), of $S=V(\Gamma)$. Hence $S\,\varepsilon\,C^2$, as claimed.

11. If the envelope $V(\Gamma)$ is replaced by the envelope $W(\Gamma)$ (cf. Section 9), then (39) becomes replaced by

$$X(s,t)=X(s)+t\{\tau(s)U_1(s)+\kappa(s)U_3(s)\} \tag{43}$$

(cf., e.g., [2], pp. 45-46). If only $\Gamma\,\varepsilon\,C^3$ (and, as always, $\kappa(s)>0$) is assumed, then, since the torsion $\tau(s)$ can be any continuous function (cf. the remark made at the end of Section 9), the continuous function (43) need not be differentiable.

Suppose therefore that $\Gamma \varepsilon C^4$. Then $\tau(s) \varepsilon C^1$ (and $\kappa(s) \varepsilon C^2$). But since $\tau(s) \varepsilon C^2$ need not hold, (43) will not in general be a function of class C^2. Accordingly, the "natural" parametrization, (43), of $S = W(\Gamma)$, where $\Gamma \varepsilon C^4$, is just a C^1-parametrization, provided that the inequality (40) (which, if

$$\beta = \beta(s,t) = \kappa(s) + t\{\tau'(s)\kappa(s) - \tau(s)\kappa'(s)\}, \tag{44}$$

turns out to be equivalent to $\beta \neq 0$) is satisfied. Nevertheless, $W(\Gamma) \varepsilon C^2$ (as long as $\beta \neq 0$).

In fact, if (38) is used in the same way as in Section 10, it follows from (43) that what now correspond to (42) and (41) are

$$[X_s, X_t] = -\beta U_2 \tag{45}$$

and (44), respectively, Hence, if the "path" of the points (s,t) at which (44) vanishes is excluded from the surface $W(\Gamma)$, then $W(\Gamma) \varepsilon C^2$, where $\Gamma \varepsilon C^4$, follows from (45) in the same way as $V(\Gamma) \varepsilon C^2$, where $\Gamma \varepsilon C^3$, followed from (42) in Section 10. It is also seen that, as long as $\beta(s,t)$ does not vanish, $n = 4$ can be replaced by any $n \geqq 4$ in the assertion $W(\Gamma) \varepsilon C^{n-2}$, if $\Gamma \varepsilon C^n$ is the assumption.

12. Let $S \varepsilon C^*$ mean that $S \varepsilon C^1$ and that S possesses a continuous mean curvature (in the sense of Section 2). Thus $S \varepsilon C^2$ implies that $S \varepsilon C^*$ but the converse is not true (Sections 4-5). This situation leads to various unanswered questions of which only three, (α), (β) and (γ) below, will be mentioned.

(α) If an $S \varepsilon C^*$ is a convex surface (in the sense that its tangent planes, which exist, since $S \varepsilon C^1$ by virtue of $S \varepsilon C^*$, support S), must S be of bounded curvature K in the sense of A. D. Alexandroff [1], Chap. XI and pp. 491-514 (and must this K satisfy (5))? It is natural to raise this question since $K \geqq 0$ holds formally but the proof of (5), when based on (3)-(4), asumes that $S \varepsilon C^2$. In this connection, cf. also (+?) in [22], p. 848.

(β) If $S \varepsilon C^*$, must S possess a C^1-parametrization (1) in terms of which the first fundamental form (8) becomes isothermic (i.e., $E = G$, $F = 0$)? This question has some interest in view of [4], where H is replaced by K. An affirmative answer would be surprising, since H, in contrast to K, has little to do with the metric (8).

(γ) Does $S \varepsilon C^1$ imply $S \varepsilon C^2$ if S possesses continuous curvatures H, K and is free of "umbilical points," points at which the sign of equality holds in (5)? The answer to those variants of this question in which the indices of differentiability are raised is known to be affirmative (cf. [8], p. 128). It is understood that, in the present case, K must be thought of as defined in terms of the metric (8) of $S \varepsilon C^*$; for instance, by assuming that S is a C^1-embedding of a regular C^1-metric (in this connection, cf. also (iv?) in [22], pp. 846-848). Incidentally, it is now not clear that the sign of inequality cannot reverse itself in (5); cf. question (β). Correspondingly, (γ) has an analogue in the direction of (β).

Other problems are those in the large, such as the uniqueness statement of Christoffel and the corresponding existence question for convex surfaces $S \varepsilon C^*$. The latter question concerns the existence of a closed, strictly convex $S \varepsilon C^*$ for which H is assigned as a (positive, continued) function of the normal. The method of construction which in Part II below will be applied to another embedding problem (Weyl) supplies a counterexample only if H is allowed to have an infinity. A similar situation prevails if H is replaced by K (Minkowski).

*Remark.** Suppose that an $S \varepsilon C^2$ has a C^2-representation (27) over a domain D of the (x, y)-plane and let J be a rectifiable Jordan curve which, along with its interior $B = B(J)$, is contained in D. It was recently observed by E. Heinz (*Mathematische Annalen*, vol. 129 (1955), pp. 451-454) that

$$1/r \geqq \min_{B+J} |H|, \tag{46}$$

if J is (or surrounds) a circle of radius r.

Since the proof of (46) depends on (28), which is (10) in the case (27), and since (10) is paralleled by (12), it is natural to raise the question concerning a counterpart of (46) in which H is replaced by K. It will turn out that the answer to this question is

$$|I|/(2\pi r^2) \geqq \min_{B+J} |K|, \tag{47}$$

if $|I|$ denotes the length of I, where I is the image (on the unit sphere $|N| = 1$) of J under the Gaussian mapping $N = N(u, v)$ of S: $X = X(u, v)$.

Actually, both (46) and (47) can be generalized to the case in which

* Added November 19, 1955.

J is any rectifiable Jordan curve. In fact, if $|J|$ denotes the length of J, and $|B|$ the area of the (x,y)-domain B surrounded by J, then

$$|J|/|B| \geqq \min_{B+J} 2|H| \tag{48}$$

and

$$|I|/|B| \geqq \min_{B+J} 2|K|. \tag{49}$$

Clearly, (48) reduces to (46), and (49) to (47), if J is a circle of radius r. The proof of (48) (which is between the lines of Heinz's proof of (46)) and of (49) proceeds as follows:

Since $|[Y,Z]| \leqq |Y|\,|Z|$ and $|N| = 1$, the absolute values of the line integrals occurring in (10) and (12) are majorized by

$$\int_J |dX| = |J| \text{ and } \int_J |dN| = |I|, \tag{50}$$

respectively. On the other hand, since (1) is given in the form (27), the relation (10) simplifies to (28), and (12) to a relation similar to (28) (with $2H$ replaced by $2K$ in the $dxdy$-integral). Accordingly, (10), (12) and the majorants (50) of the line integrals lead to

$$\left|\int_B 2H dxdy\right| \leqq |J|, \qquad \int_B 2K dxdy \,| \leqq |I|. \tag{51}$$

Clearly, (48) and (49) follow from (51).

Heinz points out (loc. cit., Satz 2) that (46) can be combined with (5) if S satisfies the convexity assumption $K > 0$. Under the same assumption, (48), too, can be combined with (5) and leads to

$$(\tfrac{1}{2}|J|/|B|)^2 \geqq \min_{B+J} K > 0 \tag{52}$$

(whereas the classical isoperimetric inequality, being the relation $|J|/|B|^{\frac{1}{2}} \geqq (4\pi)^{\frac{1}{2}}$, cannot be combined with (48) or (52), at least not directly). But (52) is of a type quite different from (49) and, in contrast to (52), neither (49) nor (48) asumes that S is convex.

II. Embedding and Gaussian Curvature.

1. In general terms, Weyl's embedding problem can be formulated as follows: On a two-dimensional manifold $\mathfrak{S}$ which is of the topological type of a sphere, there is assigned a metric μ which (in sense to be specified)

has a positive curvature. The problem consists in finding in the Euclidean (x, y, z)-space a surface Σ which is topologically equivalent to Θ and which realizes μ on Σ by virtue of the topological correspondence between Θ and Σ.

From the point of view of differentiability assumptions, there is today a considerable gap between what is known to be true and what is known to be false concerning the existence of a Σ (cf. [11] and [7], respectively). Corresondingly, the assumptions of smoothness on μ under which the existence of a Σ of appropriate smoothness is assured today, appear to be quite restictive from the point of view of *local* embeddings (in this regard, cf. [22]; in the existence proof given in [11], the strictness of the assumptions placed on μ is originated, as in [17], by the necessity of appealing to Weyl's inequality (cf. [23]) in the treatment of the *non-local* problem).

No such analytical restrictions on μ, and no corresponding complications in the proof, arise in A. D. Alexandroff's approach to the problem (cf. [1], Chap. VII; cf. Chap. XI). This approach, like Minkowski's treatment of his embedding problem, obtains Σ as a limit of closed convex polyhedrons $\Pi_1, \Pi_2, \cdots$. Correspondingly, since the strength of the limit process $\Pi_i \to \Sigma$ is not under control from the point of view of differentiability (as a matter of fact, Π_i itself is not a differentiable surface), no assertion of smoothness can result for Σ.

The following comments have the purpose of exhibiting the intrinsic necessity of not claiming any smoothness for Σ (at least when the curvature $K > 0$ of μ is allowed to be ∞ at a point of Θ; except for this point, μ will be a regular analytic Riemannian metric).

2. Let D be a domain, say the circle $D = D_a$: $u^2 + v^2 < a^2$, in a parameter plane (u, v), and let there be given on D a metric ds^2, that is, a quadratic form (8) (but no function X), with coefficients E, F, G which are continuous on D and satisfy the condition $g > 0$, where $g = (EG - F^2)^{\frac{1}{2}} > 0$. Then ds^2 will be called a continuous metric (on D). By a C^1-embedding S of ds^2 in the X-space, where $X = (x, y, z)$, is meant a surface S: $X = X(u, v)$, where $X(u, v)$ is a vector function of class C^1 satisfying (8). If $X(u, v)$ is a function of class C^n, then the surface S is called a C^n-embedding of ds^2.

Let $B = B_a$ denote the domain which results if the point $(u, v) = (0, 0)$ is removed from the circle $D = D_a$, so that B_a: $0 < u^2 + v^2 < a^2$, and let ds^2 be a metric which is continuous on D_a: $u^2 + v^2 < a^2$, with coefficients E, F, G which are of class C^2 (or, for that matter, regular analytic) on B_a and possess a curvature $K = K(u, v)$ which tends to ∞ as $u^2 + v^2 \to 0$ (so that $K > 0$ on B_a, if a is small enough). It will be shown that, roughly speaking,

such a continuous metric ds^2 on D_a can be chosen in such a way that for no positive $c(<a)$ will the metric ds^2 on D_c possess any *convex* C^1-embedding S (even though there exists on B_c an analytic $K(u,v)$ having a positive lower bound, and even though E, F, G are continuous on D_c or on the closure of B_c).

Since $K>\text{const.}>0$, the italicized proviso, that of restricting the C^1-embeddings S to be convex, seems to be redundant (the more so as the metric is regular analytic on B_c). But as matters seem to stand today, this conclusion is valid only if the surfaces S admitted are assumed to be C^2-embeddings, rather than just C^1-embeddings (in this regard, cf. (iv?) and (+?) in [22], p. 848.* As a matter of fact, the proof to be given below will assume that the C^1-embedding S of the continuous metric ds^2 is a C^2-embedding near all those points (u,v) at which the functions E, F, G are not of class C^2 (and these functions will be regular analytic except at a single point, a point at which they will remain continuous). All such embeddings S turn out to be convex (as a matter of fact, strictly convex) by virtue of $K>0$; cf. the proof in Section *4* below.

3. Let $E=1$ and $F=0$, hence $g=G^{\frac{1}{2}}$; so that

$$ds^2=du^2+g^2dv^2, \tag{1}$$

where $g=g(u,v)>0$. Then, at those points (u,v) at which g has continuous second derivatives, $K=K(u,v)$ is given by

$$g_{uu}+Kg=0 \tag{2}$$

(Jacobi). Define on D_a a continuous metric (1) by placing $g(0,0)=1$ at D_a-B_a and

$$g(u,v)=1+(\cos\phi)^2/\log r,\ \text{ where } u=r\cos\phi,\ v=r\sin\phi \tag{3}$$

$(r>0)$; so that $g(u,v)$ is positive and regular analytic on B_a (if $a>0$ is small enough). Incidentally, cf. (*3*) with (17) above.

Substitution of (*3*) into (*2*) shows that $K(u,v)\sim-2/(r^2\log r)$ as $r\to\infty$. This implies that $K>0$ on B_a (if $a>0$ is small enough) and that

* As will, however, be shown elsewhere, a treatment of questions (+?)-(—?) of [22], p. 848, and of the related questions, pointed out above, can be based on the results of A. D. Alexandroff [1], p. 51, concerning convex metrics.

the integral of $K(u,v)$ over B_a is ∞. Since $g(u,v)\to 1$ as $(u,v)\to(0,0)$, this is equivalent to

$$(4) \qquad \iint_{B_a} Kgdudv = \infty \qquad (K>0, g>0).$$

It will be concluded from (4) that, no matter how small $a>0$ be chosen, there cannot belong to (1) on D_a any surface S: $X=X(u,v)$ of class C^1 satisfying $ds=|dX|$, provided that, corresponding to (but, as far as present knowledge seems to go, perhaps not implied by) the fact that the coefficient g of (1) is regular analytic, and K is positive, on B_a (with $K=\infty$ at D_a-B_a), that portion, say S_0, of the C^1-surface S which corresponds to B_a is assumed to be of class C^2 (hence, strictly convex).

4. Suppose the contrary. Then, since S is a surface of class C^1, it can be assumed to be in the form S: $z=z(x,y)$, where x,y,z are Cartesian coordinates, $(u,v)=(0,0)$ corresponds to $(x,y,z)=(0,0,0)$, the function $z(x,y)$ is of class C^1 in an (x,y)-neighborhood, say V, of the point $(x,y)=(0,0)$ and, if Q denotes the latter point, the (x,y)-plane is tangent to S at Q (i.e., $z_x=0$ and $z_y=0$ at Q). In terms of the notation S_0, introduced at the end of Section 3, the surface S_0 results if the origin is removed from the surface S. Since S_0 is supposed to be of class C^2, the function $z(x,y)$ has continuous second derivatives on the (x,y)-domain $V-Q$.

According to Satz IV of Schilt [13], p. 257, the origin is the only point of S on the (x,y)-plane (if V or $a>0$ is small enough). In fact, $z(x,y)$ is of class C^2, hence the normal vector to S_0 is of class C^1, on $V-Q$, and the proof, given in [13], is such that the "gradient," considered on p. 244, need not be assumed to exist at the critical point. Accordingly, it can be assumed that $z(x,y)>0$ on $V-Q$, since $z(x,y)=0$ at Q.

Also the consideration of Schilt on pp. 250-251 of [13] (§9-§10) remain valid, since, for the proofs given loc. cit., only the existence and the continuity, but not the differentiability, of the normal to S is needed at Q. Hence, Satz V of Schilt [13], p. 257, is applicable.

This means that, if $N=N(x,y)$ denotes the (oriented) unit normal at the point (x,y) of S (where the boundary point $(0,0)$ of $S-S_0$ is included), then $N=N(x,y)$ represents a one-to-one continuous mapping of a neighborhood V of Q on the unit sphere. Let T denote the spherical image of V or S, and let $|T|$ be the area of R. Then, since T is a schlicht image of V, and since $K>0$ on $V-Q$,

$$|T| = \lim_{\epsilon\to 0} \iint_{B_a-B_\epsilon} Kgdudv$$

(Gauss). It follows therefore from (4) that $|T| = \infty$. But $|T| = \infty$ contains a contradiction. In fact, since T is a schlicht piece of the unit sphere, $|T| < 4\pi$.

5. The preceding results is of a local nature, since it deals only with a sufficiently small D_a. It is, however, clear that the case (3) of (1) can be extended from D_a to a closed, orientable (u, v)-manifold in such a way as to be of class C^∞ and of positive $K(u, v)$ on the closure of $\Sigma - D_a$, and the passage from the class C^∞ to the class of regular analyticity (except for the center of V_a) also offers no difficulty. What thus results is the situation announced in Section 2.

ADDENDUM.* Let there be given on the abstract sphere Θ a positive definite metric form ds^2, suppose that the coefficients E, F, F of ds^2 are functions of class C^1 in suitable local parameters (u, v) on Θ and that ds^2 has at every point of Θ a continuous curvature K, finally that $K > 0$ throughout. The general existence statement in Weyl's paper [17] is that, under these assumptions, the ds^2 on Θ can be realized on a (strictly) convex, closed surface Σ of class C^2 in the X-space, where $X = (x, y, z)$. This existence statement of Weyl will be referred to as (*).

Weyl was aware that he did not fully succeed in proving (*) (cf. [11], where further references are given), and it was observed in [7] that certain variants of (*), variants the truth of which one would expect if (*) is true, are certainly false. It will now be shown that (*) itself is false.

For a real $z = z(x, y)$, and on a sufficiently small neighborhood of $(x, y) = (0, 0)$, consider the implicit equation

$$z = \tfrac{1}{2}(x^2 f + y^2/f), \text{ where } f = f(z) = 2 + \sin\log(1/\log z); \tag{5}$$

so that $z(x, y) > 0$ unless $(x, y) = (0, 0)$, and $z(0, 0) = 0$. Let D be the circle $0 \leqq r < a$, and D_0 the punctured circle $0 < r < a$, where $r = (x^2 + y^2)^{\frac{1}{2}}$. According to A. D. Alexandroff [1], pp. 446-447 (where $\log w$ must be interpreted as $\log - w$ if $w < 0$), the implicit relation (5) defines over D a differentiable convex cap S: $z = z(x, y)$ which is such that, although the plane curves which represent the normal sections of S at the point $D - D_0$ fail to have curvatures,

$$S \text{ possesses a continuous } K > 0 \text{ on } D \tag{6}$$

(hence, at the point $D - D_0$ also). Clearly, S is analytic (hence $S \,\varepsilon\, C^2$) on D_0, but what was mentioned before (6) (and, in fact, (5) as it stands) shows

* Added November 19, 1955.

that $S \varepsilon C^2$ cannot hold on D. In what follows, it will be granted that, as claimed by Alexandroff, (5) defines over D a convex cap S satisfying (6).

Let every O refer to $(x,y) \rightarrow (0,0)$, i. e., to $r \rightarrow 0$. Then it is clear from (5) that $z(x,y) = O(r^2)$. Since S is convex, this implies that the derivatives $p = z_x$, $q = z_y$, which are analytic on D_0, are of the form $O(r)$ near $D - D_0$. By a standard property of functions which are derivatives of a continuous function, this implies that p and q exist $(=0)$ and are continuous at $D - D_0$. Hence $S \varepsilon C^1$ on D.

Since the Gaussian parameters (u,v) are (x,y) in the C^1-parametrization $z = z(x,y)$ of S, the (continuous) coefficients of ds^2 on D are given by (21) above. On the other hand, the argument which led to (20) is applicable, since p and q are analytic, hence of class C^1, on D_0 (though not on D) and continuous and of the form $O(r)$ at $D - D_0$ (actually, only $p = o(r^{\frac{1}{2}})$ and $q = o(r^{\frac{1}{2}})$ are needed). Hence, E, F, G are of class C^1 on D.

The situation can be summarized as follows: On D, both $S \varepsilon C^1$ and (6) hold, but $S \varepsilon C^2$ does not hold (except on D_0), although S possesses on D a C^1-parametrization in which the coefficients of the ds^2 on S are functions of class C^1. In order to conclude from this that the assertion (*) is false, it is sufficient to repeat the argument applied on p. 487 in [7].

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A VARIATIONAL METHOD IN THE THEORY OF HARMONIC INTEGRALS, II.* [1]

By Charles B. Morrey, Jr.

1. Introduction. In this part, the variational method introduced in part I [9] is applied to the study of boundary value problems for exterior differential forms on a compact Riemannian manifold $\mathfrak{M}$ with boundary $\mathfrak{B}$. The manifold is not assumed to be orientable and parallel theories are developed for even and odd forms.

We shall use the results of part I extensively and shall refer to it frequently; the words part I in such a reference will stand for that paper [9]. We retain the notations of that part except for one change used only in Sections 2 and 3 and introduce new notations as required.

In Section 2, we discuss the behavior of the G-quasi-potentials and G-potentials defined in Section 3 of part I on the part $x^n = 0$ of the boundary of a hemisphere. The behavior of certain more general quasi-potentials and potentials, satisfying a "natural boundary condition" on $x^n = 0$ is also studied. In Section 3, systems of equations in integrated form like those in Section 4 of part I are studied, particularly with reference to the differentiability properties of the solutions along $x^n = 0$. Certain approximation and boundedness theorems are also proved. The results of these two sections form the analytic basis for our results concerning the differentiability of the solutions at the boundary.

In Section 4 important preliminary material is presented: Riemannian manifolds with boundary of class C_μ^k, etc., are defined, certain results about $\mathfrak{P}_2$ forms are carried over from Section 5 of part I, the tangential and normal parts of the boundary values of forms are defined, a number of important lemmas are proved, and the Gaffney inequality [5] and theorem are proved for the closed linear subspaces $\mathfrak{P}_2^+$ of $\mathfrak{P}_2$-forms ω for which $n\omega = 0$ and $\mathfrak{P}_2^-$ of $\mathfrak{P}_2$-forms ω for which $t\omega = 0$.

The Gaffney theorem for the spaces $\mathfrak{P}_2^+$ and $\mathfrak{P}_2^-$ makes it possible to

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carry over verbatim the analysis of Section 6 of part I to each of the spaces $\mathfrak{P}_2^+$ and $\mathfrak{P}_2^-$. The differentiability theory of Section 3, together with the approximation device used in the proof of Theorem 7.1 of part I, is used to establish complete results concerning the differentiability properties of the "plus and minus potentials" and their derivatives. These results are then used to establish an orthogonal decomposition theorem $\mathfrak{L}_2 = \mathfrak{H} \oplus \mathfrak{C} \oplus \mathfrak{D}$ similar to that of Kodaira [6] given in part I in which $\mathfrak{H}$ consists of all harmonic fields in $\mathfrak{L}_2$, and $\mathfrak{C}$ and $\mathfrak{D}$ consist, respectively, of all $\mathfrak{L}_2$-forms of the form $\delta\alpha$ for α in $\mathfrak{P}_2^+$ and $d\beta$ for β in $\mathfrak{P}_2^-$. The differentiability of the projections of a given form on these spaces is discussed completely and several other interesting related theorems are proved. In this case, the manifold $\mathfrak{H}$ has infinite dimensionality, but the Friedrichs inequality [4] $D(\omega) \geqq \lambda \| \omega \|^2$ is proved to hold for all ω in $\mathfrak{C} \oplus \mathfrak{D}$. This leads directly to the existence of a certain overall potential Ω in $\mathfrak{C} \oplus \mathfrak{D}$ of forms ω in $\mathfrak{C} \oplus \mathfrak{D}$ which turns out to be related to the positive and negative potentials Ω^+ and Ω^- of ω; in fact

$$(1.1) \qquad d\Omega^+ = d\Omega, \qquad \delta\Omega^- = \delta\Omega.$$

In Section 6, we begin by deducing the results in Theorems 2, 3, and 4 of the paper [3] of Duff and Spencer very quickly from the Theorems of Section 5. Next the Dirichlet problem (tK and nK given) for harmonic *forms* (Theorem 1 of Duff and Spencer [3]), as distinct from harmonic fields, is proved including the result of Spencer [11] concerning the uniqueness of the solutions in the analytic case. The potentials Ω^+, Ω^-, and Ω of Section 5 satisfy the boundary conditions

$$(1.2) \qquad n\Omega^+ = nd\Omega^+ = 0; \quad t\Omega^- = t\delta\Omega^- = 0; \quad nd\Omega = t\delta\Omega = 0.$$

These potentials are used to obtain the results of Connor [1] concerning boundary value problems for harmonic forms K. He solves the problems (i) nK and ndK given, (ii) tK and $t\delta K$ given, and (iii) ndK and $t\delta K$ given. We solve these, roughly speaking, by showing first that there is some form ω satisfying the given boundary conditions and then defining K as ω minus the relevant potential of $\Delta\omega$. This procedure is carried through under very general conditions and complete results concerning differentiability are presented. We conclude with a remark concerning a recent extension due to Spencer [10] of the Dirichlet problem to bounded manifolds.

As was pointed out in the introduction to part I, K. O. Friedrichs had obtained some results for the case of manifolds without boundary but had not published them. He has been working almost concurrently on the case

of manifolds with boundary. All of his results are included in his paper [4]. After a detailed discussion with him of our and his results and methods, we have concluded that there are sufficient differences to warrant publication of both of our papers.

2. Potentials on hemispheres. In this section we study the G-quasi-potentials and G-potentials defined in part I [9], definition 3.4 for the case that G is a hemisphere G_R where G_R is the part of $B(0,R)$ for which $x^n < 0$. We also study certain unrestricted $G(x_0,R)$-quasi-potentials and potentials which we define below in Definition 2.2. In this and the next section, we retain the notations of Sections 3 and 4 of part I unless otherwise specified. For convenience, we denote by σ_R the part of $B(0,R)$ for which $x^n = 0$ and S_R^- the part of $\partial B(0,R)$ for which $x^n \leqq 0$. We also define $G(x,r)$ as the part of $B(x,r)$ for which $x^n < 0$ and $S^-(x,r)$ as the part of $S(x,r)$ for which $x^n \leqq 0$ and define $\Gamma(x,r)$ as the union of $G(x,r)$, its reflection $G^+(x,r)$ in $x^n = 0$, and the interior of the $(n-1)$-sphere $\partial G(x,r) \cap \partial G^+(x,r)$.

THEOREM 2.1. (i) *If* $u \in \mathfrak{P}_2$ *on* $G(x_0,R)$ *and vanishes on* $S^-(x_0,R)$, *then* $d_0[u, G(x_0,R)] \leqq C_1 R d_1[u, G(x_0,R)]$, $C_1 = 2^{\frac{1}{2}}$.

(ii) *The totality of such functions forms a closed linear manifold in* $\mathfrak{P}_2$.

Proof. (i) follows from the integration of the inequality

$$\int |u(x^n, x'_n)|^2 dx'_n = \int |u(x^n, x'_n) - u(X^n, x'_n)|^2 dx'_n \tag{2.1}$$
$$\leqq (x^n - x_0^n + R) D_2[u, G(x_0,R)],$$
$$X^n = x_0^n - (R^2 - |x'_n - x'_{0n}|^2)^{\frac{1}{2}}$$

with respect to x^n. (ii) follows from Theorem 2.12, part I.

Definition 2.1. We define the space $\mathfrak{P}^*_2$ on $G(x_0,R)$ to consist of all u in $\mathfrak{P}_2$ which vanish on $S^-(x_0,R)$ with inner product given by

$$((u,v))^* = \int_{G(x_0,R)} u_{x^\alpha} v_{x^\alpha} dx$$

and norm $\| u \|^* = ((u,u))^{\frac{1}{2}}$.

Remark. It is clear that $\mathfrak{P}_{20} \subset \mathfrak{P}^*_2$.

The proof of Theorem 3.3 of part I with the obvious modifications yields the following theorem:

THEOREM 2.2. *Suppose* $e = (e_1, \cdots, e_n)$ *and* f *are in* $\mathfrak{L}_2$ *on* $G(x_0, R)$. *Then there are unique solutions* u^* *and* v^* *in* $\mathfrak{P}^*_2$ *on* $G(x_0, R)$ *such that*

$$(2.2) \qquad \int_{G(x_0,R)} w_{x^\alpha}(u^*_{x^\alpha} + e_\alpha)\,dx = 0, \quad \int_{G(x_0,R)} (w_{x^\alpha} v^*_{x^\alpha} + wf)\,dx = 0$$

for all w *in* $\mathfrak{P}^*_2$ *on* $G(x_0, R)$ *and we have*

$$\| u \|^* \leqq d_0[e, G(x_0, R)], \qquad \| v \|^* \leqq C_1 R d_0[f, G(x_0, R)].$$

Definition 2.2. The solution u^* above is called the $G(x_0, R)$-*-*quasi-potential of* e and the solution v^* is called the $G(x_0, R)$-*-*potential of* f.

The following lemma of Soboleff is proved exactly as it was in part I, Lemma 3.2:

LEMMA 2.1. *Suppose* $u, \nabla u, \cdots, \nabla^p u$ *are all of class* $\mathfrak{P}_2$ *on* $G(x_0, R)$ $p \geqq [n/2]$. *Then*

$$| u(x) | \leqq a_{n^*}^{-\frac{1}{2}} R^{-n/2} \{ \sum_{j=0}^{p} d_j[u, G(x_0, R)] (2R)^j / j! + d_{p+1}[u, G(x_0, R)] 2 \cdot (2R)^{p+1}/p! \},$$

$$x \,\varepsilon\, G(x_0, R), \qquad a_{n^*} = \tfrac{1}{2} m[B(0,1)].$$

Remark. It is clear that the definitions and theorems above extend immediately to vector functions u, v, w, etc. This is true of the following theorems and definitions also.

The proof of Lemma 3.6, part I, carries over with only very minor changes to yield a proof of Lemma 2.2 below; in fact the writer originally proved the lemma for this case (see [7], pp. 130-132).

Definition 2.3. For points $x \,\varepsilon\, G(x_0, R)$, δ_x denotes the distance of x from $S^-(x_0, R)$.

LEMMA 2.2. *Suppose* $u \,\varepsilon\, \mathfrak{P}_2$ *on* $G(x_0, R)$ *and suppose*

$$d_1[u - l_{yr}, G(y, r)] \leqq L(r/\delta_y)^{\rho+\mu}, \quad 0 \leqq r \leqq \delta_y, \quad 0 < \mu < 1, \quad \rho = n/2,$$

for each $y \,\varepsilon\, G(x_0, R)$ *where* l_{yr} *is the linear function of Lemma* 3.5, *part* I, *for* u *on* $G(y, r)$. *Then* u *is of class* C^1_μ *on* $G(x_0, R)$ *and there is a constant* $C_2 = C_2(n, \mu)$ *such that*

$$| \nabla u(\xi) - \nabla u(x) | \leqq C_2 L \delta_x^{-\rho-\mu} \cdot | \xi - x |^\mu,$$

$$0 \leqq | \xi - x | \leqq \delta_x/2, \xi, x \,\varepsilon\, G(x_0, R).$$

Definition 2.4. We define the *spaces* $\mathfrak{L}_{2\lambda}$ and $C_\mu{}^0$ of functions in $\mathfrak{L}_2$ on $G(x_0, R)$ and the respective norms $|e|_\lambda$ and $|e|_\mu{}^0$ exactly as they were defined (for functions in $\mathfrak{L}_2$ on B_R) in Definition 3.5 of part I, δ_x having its significance above. We define the *spaces* $\mathfrak{P}_{20\lambda}$ and $C_{0\mu}{}^1$, and $\mathfrak{P}^*{}_{2\lambda}$ and $C^*{}_\mu{}^1$, and the corresponding norms just as the first two were defined in Definition 3.5 of part I with the obvious changes, the first two being subsets of $\mathfrak{P}_{20}$ and the latter two being subsets of $\mathfrak{P}^*{}_2$ on $G(x_0, R)$.

The writer found it convenient to replace the spaces $\mathfrak{A}_{RK}{}^0$ and $\mathfrak{A}_{0RK}{}^1$ of Section 3, part I, by other spaces which we define below in Definition 2.6 which definition necessitates the following:

Definition 2.5. By $'\nabla^p e$, $p = 1, 2, \cdots$, we mean the set of functions defined by $'\nabla^0 e = e$, $'\nabla^p e = \{e_{x^\alpha \cdots x^\gamma}\}$, $\alpha, \cdots, \gamma \leqq n-1$. We define the non-negative quantities $d^*{}_p(e, G)$ and $d^{**}{}_p(u, G)$ by

$$d^*{}_0(e, G) = d_0(e, G), \; [d^*{}_p(e, G)]^2 = \int_G [|'\nabla^p e|^2 + |'\nabla^{p-1} e_{x^n}|^2] dx, \qquad p \geqq 1,$$

$$d^{**}{}_0(u, G) = d_1(u, G), \; [d^{**}{}_p(e, G)]^2 = \int_G [|'\nabla^p \nabla u|^2 + |'\nabla^{p-1} \nabla u_{x^n}|^2] dx, \qquad p \geqq 1.$$

Definition 2.6. We define the *space* $\mathfrak{B}_{RK}{}^0$ to consist of all e (or f) which are in $\mathfrak{L}_2$ on G_R with all the $'\nabla e^p$ in $\mathfrak{P}_2$ on each G_r with $r < R$ with norm defined by

$$|e|_{RK}{}^0 = \sup[(2p)!K^p]^{-\frac{1}{2}} \cdot (R-r)^p d^*{}_p(e, G_r), \qquad p = 0, 1, 2, \cdots, 0 \leqq r < R.$$

We define the *spaces* $\mathfrak{B}^*{}_{RK}{}^1$ and $\mathfrak{B}_{RK}{}^1$ as those subsets of $\mathfrak{P}^*{}_2$ and $\mathfrak{P}_{20}$, respectively for which all the $'\nabla u^p$ and $'\nabla u_{x^n}{}^p \varepsilon \mathfrak{P}_2$ on each G_r with $r < R$, the norm being defined by

$$\|u\|^*{}_{RK}{}^1, \|u\|_{0RK}{}^1 = \sup[(2p)!K^p]^{-\frac{1}{2}} \cdot (R-r)^p d^{**}{}_p(u, G_r), \qquad 0 \leqq r < R, p = 0, 1, 2, \cdots.$$

LEMMA 2.3. *If $u \varepsilon \mathfrak{B}^*{}_{RK}{}^1$ or to $\mathfrak{B}_{0RK}{}^1$, then all the $'\nabla u^p$ and $'\nabla u^p{}_{x^n}$ are continuous in x on G_R and analytic in x'_n for each x^n, $-R < x^n \leqq 0$.*

Proof. Suppose $-R < a < b \leqq 0$, $r^2 + a^2 < R^2$. Let v be any one of the functions above. Since v is of class $\mathfrak{P}_2$ on any $G_{r'}$ with $r' < R$, we have

$$\int_{\sigma_r} |\bar{v}(b, x'_n) - \bar{v}(a, x'_n)|^2 dx'_n < (b-a) \int_a^b [\int_{\sigma_r} |\bar{v}_{x^n}(x^n, x'_n)|^2 dx'_n] dx^n,$$

so that $\int_{\sigma_r} |\bar{v}(x^n, x'_n)|^2 dx'_n$, $r < R$, is *uniformly bounded* for $a \leqq x^n \leqq 0$ if $a^2 + r^2 < R^2$. Then Soboleff's lemma 2.1 in $(n-1)$ dimensions implies *uniform* (independent of x^n) bounds for all the $'\nabla^p u$ and $'\nabla^p u_{x^n}$ on any G_r with $r < R$ which insure their analyticity in the variables x'_n for each x^n. Since each is also absolutely continuous in x^n for almost all x'_n, the continuity in x follows from the uniform continuity (with respect to x^n) in the variables x'_n.

The following symmetry lemma is useful in the proof of our main Theorem 2.4:

LEMMA 2.4. *Suppose that e and $f \varepsilon \mathfrak{L}_2$ on $G(x_0, R)$, suppose that $u_0 = Q_{R0}(e)$, $v_0 = P_{R0}(f)$, $u^* = Q^*_R(e)$, $v^* = P^*_R(f)$, and suppose that U_0, V_0, U^*, V^*, E_0, F_0, E^*, and F^* are defined on $\Gamma(x_0, R)$ to be equal on $G(x_0, R)$ to the corresponding small lettered functions and to be defined on $G^+(x_0, R)$ by*

$$
\begin{aligned}
&U_0(x^n, x'_n) = -u_0(-x^n, x'_n), \quad U^*(x^n, x'_n) = u^*(-x^n, x'_n),\\
(2.3)\qquad &E_{0n}(x^n, x'_n) = e_n(-x^n, x'_n), \quad E^*_n(x^n, x'_n) = -e_n(-x^n, x'_n),\\
&E_{0\alpha}(x^n, x'_n) = -e_\alpha(-x^n, x'_n), \quad E^*_\alpha(x^n, x'_n) = e_\alpha(-x^n, x'_n),\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \alpha = 1, \cdots, n-1,
\end{aligned}
$$

with formulas for V_0 and F_0 like those for U_0 and formulas for V^* and F^* like those for U^*. Then $U_0 = Q_R(E_0)$, $V_0 = P_R(E_0)$, $U^* = Q_R(E^*)$, $V^* = P_R(F^*)$. Here Q_{0R}, P_{0R}, Q^*_R, and P^*_R refer to $G(x_0, R)$ and Q_R and P_R to $\Gamma(x_0, R)$.

Proof. All of these are proved in a similar way so we shall prove only the first. We note that ∇U_0 and ∇U^* are related to ∇u_0 and ∇u^* as E_0 and E^* are to e and e, respectively. Let w be any function in $\mathfrak{P}_{20}$ on $\Gamma(x_0, R)$; obviously (since $u_0 = v_0 = 0$ on $x^n = 0$), U_0, V_0, U^*, $V^* \varepsilon \mathfrak{P}_{20}$ on $\Gamma(x_0, R)$. We write $w = w_1 + w_2$ where w_1 is odd in x^n like U_0 and w_2 is even in x^n like U^*. Then $\nabla U_0 \cdot \nabla w_1$ and $\nabla w_1 \cdot E_0$ are even in x^n while $\nabla w_2 \cdot \nabla U_0$ and $\nabla w_2 \cdot E_0$ are odd; clearly $w_1 = 0$ on $\partial G(x_0, R)$. Clearly, then,

$$
\int_{\Gamma(x_0,R)} w_{x^\alpha}(U_{0x^\alpha} + E_{0\alpha})\,dx = 2\int_{G(x_0,R)} w_{1x^\alpha}(u_{0x^\alpha} + e_\alpha)\,dx = 0.
$$

We now list a few additional properties of functions in the various spaces; the analog of (ii) should have been included in Theorem 3.5 of part I.

THEOREM 2.3. (i) *There is a constant* $C_3(n,\mu)$ *such that*

$$(2.4) \qquad |e(x)| \leqq C_3 \cdot |e|_\mu{}^0 \delta_x{}^{-\rho}, \qquad x \varepsilon G(x_0, R),$$

for any $e \varepsilon C_\mu{}^0$ *on* $G(x_0, R)$.

(ii) *There is a constant* $C_4(n,\mu)$ *such that if* $e \varepsilon C_\mu{}^0$ *on* $G(x_0,R)$, *then* $e \varepsilon \mathfrak{L}_{2\lambda}$ *for any* $\lambda \leqq \rho$ *on* $G(x_0,R)$ *with* $|e|_\lambda \leqq C_4 |e|_\mu{}^0$.

(iii) *There is a constant* $C_5(n,\mu)$ *such that if* $u \varepsilon C_{0\mu}{}^1$ *or to* $C^*{}_\mu{}^1$ *on* $G(x_0, r_0)$, *then* u *and* $\nabla u \varepsilon C_\mu{}^0$ *there and*

$$|\nabla u|_\mu{}^0 \leqq \| u \|_\mu{}^1, \quad |u|_\mu{}^0 \leqq C_5 \cdot R \cdot \| u \|_\mu{}^1, \quad \| u \|_\mu{}^1 = \| u \|_{0\mu}{}^1 \text{ or } \| u \|^*{}_\mu{}^1.$$

(iv) *If* $u \varepsilon \mathfrak{B}_{0RK}{}^1$ *or* $\mathfrak{B}^*{}_{RK}{}^1$, *then* u *and* $\nabla u \varepsilon \mathfrak{B}_{RK}{}^0$ *and*

$$|\nabla u|_{RK}{}^0 \leqq \| u \|'_{RK}, \quad |u|_{RK}{}^0 \leqq [C_1 + (2K)^{-\frac{1}{2}}] R \cdot \| u \|'_{RK},$$
$$\| u \|'_{RK} = \| u \|_{0RK}{}^1 \text{ or } \| u \|^*{}_{RK}{}^1.$$

The proofs of (i), (iii), and (iv) are very similar to the corresponding parts of Theorem 3.5, part I. (ii) follows easily by squaring (2.4), integrating over $G(y,r)$, and using the finiteness of $|e| \leqq |e|_\mu{}^0$.

THEOREM 2.4. *The transformations* Q_{R0} *and* P_{R0} ($Q^*{}_R$ *and* $P^*{}_R$) *are bounded linear operators from* $\mathfrak{L}_{2\lambda}$ *to* $\mathfrak{P}_{20\lambda}(\mathfrak{P}^*{}_{2\lambda})$ *for* $0 < \lambda < \rho$, *from* $C_\mu{}^0$ *to* $C_{0\mu}{}^1(C^*{}_\mu{}^1)$ *for* $0 < \mu < 1$, *and from* $\mathfrak{B}_{RK}{}^0$ *to* $\mathfrak{B}_{0RK}{}^1(\mathfrak{B}^*{}_{RK}{}^1)$ *for any* $K > e$. *The transformation* $P_R(P^*{}_R)$ *is also a bounded linear operator from* $\mathfrak{L}_{2,\rho-1+\mu}$ *to* $C_{0\mu}{}^1(C^*{}_\mu{}^1)$ *for* $0 < \mu < 1$. *There are constants* C_6, C_7, C_8, C_9, $C^*{}_6$, $C^*{}_7$, $C^*{}_8$, *and* $C^*{}_9$ *with* $C_k = C_k(n,\mu)$ *and* $C^*{}_k = C^*{}_k(n,\mu)$, $k = 6, 7, 8$, *and* $C_9 = C_9(K,n)$, $C^*{}_9 = C^*{}_9(K,n)$ *such that*

$$\| P_{R0}(f) \|_{0\lambda} \leqq C_6 R |f|_\lambda, \qquad \| P^*{}_R(f) \|^*{}_\lambda \leqq C^*{}_6 R |f|_\lambda,$$

$$\| P_{R0}(f) \|_{0\mu}{}^1 \leqq C_7 R |f|_\mu{}^0, \qquad \| P^*{}_R(f) \|^*{}_\mu{}^1 \leqq C^*{}_7 R |f|_\mu{}^0,$$

$$\| P_{R0}(f) \|_{0\mu}{}^1 \leqq C_8 R |f|_{\rho-1+\mu}, \qquad \| P^*{}_R(f) \|^*{}_\mu{}^1 \leqq C^*{}_8 R |f|_{\rho-1+\mu},$$

$$\| P_{R0}(f) \|_{0RK}{}^1 \leqq C_9 R |f|_{RK}{}^0, \qquad \| P^*{}_R(f) \|^*{}_{RK}{}^1 \leqq C^*{}_9 R |f|_{RK}{}^0.$$

Proof. We use the notation of Lemma 2.4. We see that the first results for Q_{R0} and $Q^*{}_R$ and the first three results for P_{R0} and $P^*{}_R$ follow from that lemma, Theorems 2.2 and 2.3, and Theorems 3.5 and 3.6 of part I. Using also Lemma 3.3 and the very last part of the proof of Theorem 3.6, both of part I, we see that

$$\nabla V_0 = Q_R(G_0) + H_0, \quad \nabla V^* = Q_R(G^*) + H^*, \quad G_{0\gamma\alpha} = \delta_{\gamma\alpha} F_0, \quad G^*{}_{\gamma\alpha} = \delta_{\gamma\alpha} F^*.$$

$$L_2(H_0, B_R), L_2(H^*, B_R) \leqq Z_1{}^2(n) R^2 L_2(f, G_R).$$

We may therefore deduce the last statements for P_{R0} and $P^*{}_R$ from the symmetry lemma (note the symmetry properties of G_0), Theorem 2.3, and the last ($\mathfrak{B}_{RK}{}^0$) results for both Q_{R0} and $Q^*{}_R$.

In order to prove the last results for Q_{R0} and $Q^*{}_R$, we begin by applying Lemma 3.3 of part I *repeatedly with* $\gamma \leqq n-1$ to the functions U_0 and U^*; it is clear from the proof of that lemma that this procedure is valid. Exactly as in the proof of (iii) in Theorem 3.6 of part I, we obtain (going back to G_R).

$$(2.5) \quad D_2('\nabla^p u, G_r) \leqq 2(1 - e_0/K)^{-1}(2p)! \cdot K^p \cdot L^2 \cdot (R - r)^{-2p}, \quad L = |\, e\,|_{RK}{}^0,$$
$$u = u_0 \text{ or } u^*, \quad 0 \leqq r < R, \quad p = 0, 1, 2, \cdots, e_0 = 2.718 \cdots.$$

(2.5) yields bounds for all the derivatives of u involving at most one differentiation with respect to x^n. But on spheres interior to G_R, we may also apply Lemma 3.3 of part I *once* with $\gamma = n$ to each $'\nabla^p u$ and conclude that all the $'\nabla^p u_{x^n} \,\varepsilon\, \mathfrak{P}_2$ on such spheres. But then we obtain the equations

$$(2.6) \qquad '\nabla^p u_{x^n x^n} = -\sum_{\alpha=1}^{n-1} ('\nabla^p u_{x^\alpha x^\alpha} + '\nabla^p e_{\alpha x^\alpha}) - '\nabla^p e_{n x^n}$$

which shows that all the $'\nabla^p u_{x^n} \,\varepsilon\, \mathfrak{P}_2$ on each G_r with $r < R$, and, together with (2.5), yields the desired bounds on the $d^{**}{}_p(u, G_r)$.

The proofs of the second results for Q_{R0} and $Q^*{}_R$ are practically identical with the proof of (ii) of Theorem 3.6, part I, and proceed as follows: We note that e satisfies

$$(2.7) \quad \begin{gathered} d_0[e - e(y_0), G(y_0, r)] \leqq L(r/\delta_{y_0})^{\rho+\mu}, \quad |\, e(y_0)| \leqq Z_2(n, \mu) \cdot L \cdot \delta_{y_0}{}^{-\rho}, \\ L = Z_3(n, \mu) \cdot |\, e\,|_\mu{}^0, \quad y_0 \,\varepsilon\, G_R. \end{gathered}$$

We shall show in both cases that (2.7) implies that

$$(2.8) \qquad d_1[u - l_{y_0 r}, G(y_0, r)] \leqq Z_4(n, \mu) \cdot L \cdot (r/\delta_{y_0})^{\rho+\mu}, \qquad 0 \leqq r \leqq \delta_y;$$

the result will then follow from Lemma 2.2 since y_0 is arbitrary.

We first consider $Q^*{}_R$ and let y_0 be fixed. For each s, $0 \leqq s \leqq 1$, define $\psi(s) = L^{-1}$ times the sup of the left side of (2.8) ($u = u^*$) for all $G(x_0, R)$ containing y_0, all e in $\mathfrak{L}_2$ and satisfying (2.7) (for all r) on $G(x_0, R)$, and $r = s\delta_{y_0}$. Then choose an arbitrary $G(x_0, R)$ containing y_0, an arbitrary

e in $\mathfrak{L}_2$ and satisfying (2.7) there (for all r), choose $0 < r \leqq r_0 \leqq \delta_{y_0}$, and let $u^* = Q^*_R(e)$. We write

$$(2.9) \qquad u^* = u^*_{y_0r_0} + h^*_{y_0r_0}, \qquad u^*_{y_0r_0} = v^*_{y_0r_0} - e_n(y_0)k^*_{r_0y_0}$$

where $u^*_{y_0r_0}$, $v^*_{r_0y_0}$, and $k^*_{r_0y_0}$ are the $G(y_0, r_0)$-*-quasi-potentials of e, $e - e(y_0)$, and e_0 where $e_{0\alpha} = -\delta_{\alpha n}$, respectively, and $h^*_{y_0r_0}$ is the harmonic function $= u$ on $S^-(y_0, r_0)$. From the symmetry lemma, we see (by comparing U^* and $U^*_{y_0r_0} = 0$ on $\partial\Gamma(y_0, r_0)$, etc.) that

$$(2.10) \qquad D_2[u^*, G(y_0, r_0)] = D_2[u^*_{y_0r_0}, G(y_0, r_0)] + D_2[h^*_{y_0r_0}, G(y_0, r_0)].$$

From (2.7), we conclude that

$$(2.11) \qquad d_0[e, G(y_0, r)] \leqq Z_5(n, \mu) \cdot (r/\delta_{y_0})^{\rho}.$$

Thus, using the symmetry, (2.10), (2.11), the fact that $B(y_0, r_0) \subset \Gamma(y_0, r_0)$, and the first result for Q^*_R, we see that

$$(2.12) \qquad d_1[h^*_{y_0r_0}, B(y_0, r_0)] \leqq Z_6(n, \mu, \epsilon) \cdot (r_0/\delta_y)^{\rho-\epsilon}, \qquad 0 < \epsilon \leqq \rho.$$

Finally, it is easy to see that

$$(2.13) \quad k^*_{y_0r_0}(x) = x^n + k^{**}_{y_0r_0}(x),\ k^{**}_{y_0r_0x^n}(0, x'_n) = 0,\ \nabla k^{**}_{y_0r_0}(0, y'_{0n}) = 0,$$

$k^{**}_{y_0r_0}$ being the harmonic function on $\Gamma(y_0, r_0)$ which $= |x^n|$ on $\partial\Gamma(y_0, r_0)$.

Now, clearly,

$$(2.14) \quad \begin{aligned} d_1[u^* - l_{y_0r}, G(y_0, r)] &\leqq d_1[v^*_{y_0r_0} - l_r^1, G(y_0, r)] \\ &+ d_1[h^*_{y_0r_0} - l_r^2, G(y_0, r)] + d_1[e_n(y_0) \cdot k^*_{y_0r_0} - l_r^3, G(y_0, r)] \end{aligned}$$

where the various l's have their usual significance. From our definition of ψ, we have

$$(2.15) \qquad d_1[v^*_{y_0r_0} - l_r^1, G(y_0, r)] \leqq L \cdot (r_0/\delta_y)^{\rho+\mu} \cdot \psi(r/r_0)$$

since $e - e(y_0)$ satisfies (2.7) on $G(y_0r_0)$ with L replaced by $L \cdot (r_0/\delta_y)^{\rho+\mu}$. Using (2.12) and Lemma 3.4 of part I, we obtain

$$(2.16) \quad \begin{aligned} d_1[h^*_{y_0r_0} - l_r^2, G(y_0, r)] &\leqq d_1[h^*_{y_0r_0} - l_{y_0}^4, G(y_0, r)] \\ &\leqq d_1[h^*_{y_0r_0} - l_{y_0}^4, B(y_0, r)] \\ \leqq (r/r_0)^{\rho+1} d_1[h^*_{y_0r_0} - l_{y_0}^4, B(y_0, r_0)] &\leqq (r/r_0)^{\rho+1} d_1[h^*_{y_0r_0}, B(y_0, r_0)] \\ \leqq Z_6(n, \mu, \epsilon)(r_0/\delta_y)^{\rho-\epsilon}(r/r_0)^{\rho+1} & \end{aligned}$$

where $l_{y_0}{}^4$ is tangent to $h^*{}_{y_0r_0}$ at y_0. Finally, from (2.13), we obtain

$$
\begin{aligned}
& d_1[e_n(y_0)\cdot k^*{}_{y_0r_0} - l_r{}^3, G(y_0,r)] \leqq |e_n(y_0)|\cdot d_1[k^{**}{}_{y_0r_0} - l_{y_0}{}^5, G(y_0,r)] \\
(2.17) \qquad & \leqq (r/r_0)^{\rho+1}\cdot |e_n(y_0)|\cdot d_1[k^{**}{}_{y_0r_0}, \Gamma(y_0,r_0)] \\
& \leqq (2a_n)^{\frac{1}{2}}\cdot Z_2\cdot L\cdot (r_0/\delta_{y_0})^{\rho}\cdot (r/r_0)^{\rho+1}, \qquad a_n = m[B(0,1)].
\end{aligned}
$$

Combining (2.14) to (2.17), setting $s = r/\delta_{y_0}$, $t = r_0/\delta_{y_0}$, combining the last two terms, and using the arbitrariness of $G(x_0, R)$ and e, we obtain

$$\psi(s) \leqq t^{\rho+\mu}\psi(s/t) + Z_7(n,\mu,\epsilon)t^{-\epsilon}(s/t)^{\rho+1}.$$

The result for $Q^*{}_R$ now follows from the last part of the proof of (ii), Theorem 3.6, part I.

The proof for Q_R is the same except that we may take $u_{0r_0y_0} = v_{0r_0y_0}$ in (2.9) and (2.10) follows since $u_{0r_0y_0} = 0$ on $\partial G(y_0, r_0)$.

3. Regularity properties along $x^n = 0$. In this section, we prove certain differentiability properties of the solutions of differential equations in integral form of the type

$$
(3.1) \qquad \int_{G_R} \{w_{,\alpha}{}^i(a_{ij}{}^{\alpha\beta}u_{,\beta}{}^j + b_{ij}{}^{\alpha}u^j + e_i{}^{\alpha}) + w^i(b^*{}_{ij}{}^{\alpha}u_{,\alpha}{}^j + c_{ij}u^j + f_i)\}dx = 0, \qquad i = 1,\cdots,N,
$$

for all w in $\mathfrak{P}^*{}_{20}$ where $\mathfrak{P}^*{}_{20}$ consists of all vectors w in $\mathfrak{P}_2$ with

$$
(3.2) \qquad w^i = 0 \text{ on } S_R^-,\ i = 1,\cdots,N, \qquad w^i = 0 \text{ on } \sigma_R,\ i = 1,\cdots,k, \quad 0 \leqq k \leqq N.
$$

In general, we shall also assume that

$$
(3.3) \qquad u^i = 0 \text{ on } \sigma_R, \quad i = 1,\cdots,k.
$$

We assume that

$$
(3.4) \qquad a(0) = a_0, \qquad a_{0ij}{}^{\alpha\beta} = \delta^{\alpha\beta}\cdot\delta_{ij}
$$

and also that if x_0 is any point in G_R, there is a linear transformation of E^n into itself so that the new $'a(x_0) = a_0$. We assume at least that the a's are Lipschitz with b, b^*, and c bounded and measurable and e and $f \varepsilon \mathfrak{L}_2$. Additional assumptions will be made about the coefficients as desired. We do not assume that b and b^* are related in any way. The integer k above will be held fixed throughout this section.

As in part I, we assume that $u \varepsilon \mathfrak{P}$, and we wish to conclude further

differentiability properties of u by making further assumptions about the coefficients. As before, we write

$$u = u_0 + H \tag{3.5}$$

where now

$$u_0 \varepsilon \mathfrak{P}^*_{20}, \; H^i = 0, \; i = 1, \cdots, k, \; H_{,n}{}^i = 0, \; i = k+1, \cdots, n \text{ on } \sigma_R. \tag{3.6}$$

The harmonic vector H may be found by first extending u to B_R by

$$\begin{aligned} u^i(-x^n, x'_n) &= -u^i(x^n, x'_n), \quad i = 1, \cdots, k, \\ u^i(-x^n, x'_n) &= u^i(x^n, x'_n), \quad i = k+1, \cdots, n \end{aligned} \tag{3.7}$$

and then choosing H to coincide with u on ∂B_R; it is easy to see from the uniqueness that the H^i also satisfy (3.7) and hence (3.6). Then we note that

$$\int_{G_R} w_{,\alpha}{}^i a_{0ij}{}^{\alpha\beta} H_{,\beta}{}^j dx = 0 \text{ for all } w \varepsilon \mathfrak{P}^*_{20}. \tag{3.8}$$

Then reasoning as in Section 4 of part I, we see that

$$\begin{aligned} &u_0 = Tu_0 + V + W, \quad Tu_0 = Q_R[e_0(u_0)] + P_R[f_0(u_0)], \\ &V = Q_R[e_0(H)] + P_R[f_0(H)], \quad W = Q_R(e) + P_R(f) \\ &e_0(\phi) = (a - a_0) \cdot \nabla\phi + b \cdot \phi, \quad f_0(\phi) = b^* \cdot \nabla\phi + c \cdot \phi \end{aligned} \tag{3.9}$$

where $Q_R(e)$ and $P_R(e)$ are now the vectors $[Q_R{}^i(e)]$ and $[P_R{}^i(e)]$ where

$$\begin{aligned} Q_R{}^i(e) &= Q_{0R}(e_i{}^\alpha) \text{ and } P_R{}^i(f) = P_{0R}(f_i), \quad i = 1, \cdots, k, \\ Q_R{}^i(e) &= Q^*_R(e_i{}^\alpha) \text{ and } P_R{}^i(f) = P^*_R(f_i), \quad i = k+1, \cdots, N. \end{aligned} \tag{3.10}$$

The proof of Theorem 3.1 below is just like that of Theorem 4.1 of part I; the spaces and norms mentioned are defined in the obvious way:

THEOREM 3.1. (i) *If a is Lipschitz and b, b^*, and c are bounded and measurable with, say,*

$$\begin{aligned} &|a(x_1) - a(x_2)| \leqq L_1 \cdot |x_1 - x_2|, \\ &|b(x)| \leqq L_2, \quad |b^*(x)| \leqq L^*_2, \quad |c(x)| \leqq L_3, \end{aligned} \tag{3.11}$$

then T is an operator on $\mathfrak{P}^*_{20}$ and on $\mathfrak{P}^*_{20\lambda}$, $\lambda = \rho - 1 + \mu$, $0 < \mu < 1$, where

$$\| T \|^*_0 \leqq C_{10}(L_1, L_2, L^*_2, L_3, R_0) R,$$

$$\| T \|^*_{0\lambda} \leqq C_{11}(n, \mu, L_1, L_2, L^*_2, L_3, R_0) \cdot R, \; R \leqq R_0$$

$$C_{10} = L_1 + C_1(L_2 + L^*_2) + C_1{}^2 R_0 L_3.$$

(ii) *If, also* $b \varepsilon C_\mu{}^0$ *with*

$$|b(x_1) - b(x_2)| \leqq L_4 \cdot |x_1 - x_2|^\mu, \quad 0 < \mu < 1, \tag{3.12}$$

then T *is an operator on* $C^*{}_{0\mu}{}^1$ *and*

$$\| T \|^*{}_{0\mu}{}^1 \leqq C_{12}(n, \mu, L_1, \cdots, L_4, L^*{}_2, R_0) \cdot R, R \leqq R_0.$$

(iii) *If, also, the coefficients are all analytic on* G_{R_0} *with*

$$|\nabla^p a| \leqq A \cdot p!F^p, \quad |\nabla^p b| \leqq B \cdot p!F^p, \quad |\nabla^p b^*| \leqq B^* p!F^p, \quad |\nabla^p c| \leqq C p!F^p,$$

on G_{R_0} *for* $p = 0, 1, 2, \cdots$, *then* T *is an operator on* $\mathfrak{B}^*{}_{0RK}{}^1$ *with*

$$\| T \|^*{}_{0RK}{}^1 \leqq C_{13}(n, K, A, B, B^*, C, F, R_0) \cdot R \text{ if } R \leqq R_0, K > e = 2.718 \cdots.$$

COROLLARY. *If* R *is so small that* $\| T \|^*{}_0 < 1$, *then there exists a unique solution* u *of* (3.1), *in* $\mathfrak{P}_2$ *on* G_R *which coincides on* $S_R{}^-$ *with any given function* u^* *in* $\mathfrak{P}_2$ *on* G_R *and satisfies* (3.3).

For we may write (3.5) where H is determined by (3.6) and then solve (3.9) for u_0 in $\mathfrak{P}^*{}_{20}$.

THEOREM 3.2. *Suppose* a, b, b^*, *and* c *satisfy the hypotheses of Theorem* 3.1(i), *suppose* R *is small enough so that*

$$R(L_1 + C_1 L^*{}_2) = A < 1, \tag{3.13}$$

suppose u, e, *and* $f \varepsilon \mathfrak{L}_2$ *on* G_R, *and suppose* $u \varepsilon \mathfrak{P}_2$ *and satisfies* (3.1) *and* (3.3) *on each* G_r *with* $r < R$. *Then*

$$d_1(u, G_r) \leqq d_1(U_0, G_R) + (2e_0)^{\frac{1}{2}} K d_0(U, G_R) \cdot (R - r)^{-1}, \quad (e_0 = 2.718 \cdots) \tag{3.14}$$

where

$$\begin{aligned} d_1(U_0, G_R) &\leqq (1 - A)^{-1}[d_0(e, G_R) + GRd_0(f, G_R) \\ &\qquad + (L_2 + C_1 R L_3) d_0(u, G_R)] \\ d_0(U_0, G_R) &\leqq C_1 R d_1(U_0, G_R), \quad d_0(U, G_R) \leqq d_0(u, G_R) + d_0(U_0, G_R), \\ K &\leqq 1 + AR \cdot (1 - A)^{-1}. \end{aligned} \tag{3.15}$$

Proof. The proof is identical with that of Theorem 4.2 of part I with G_r replacing B_r, $\mathfrak{P}^*{}_{20}$ replacing $\mathfrak{P}_{20}$, $S_r{}^-$ replacing S_r, etc. down to the step (4.16) where

$$D_2(U, G_r) \leqq K D_2(H_r, G_r). \tag{3.16}$$

Now, if we extend U and H_r to B_r and B_R by (3.7), we see that (3.16) holds with G_r replaced by B_r. The result follows from Theorem 3.4 of part I.

The proof of the following theorem is identical with that of Theorem 4.3 of part I except for the obvious changes (B_R replaced by G_R, numbering of equations, $\mathfrak{P}_2$ replaced by $\mathfrak{P}^*_{20}$, etc.):

THEOREM 3.3. *Suppose a, b, b^*, and c and a_p, b_p, b^*_p, and c_p satisfy the hypotheses of Theorem* 3.1(i) *uniformly in p, suppose R is small enough so that $C_{10}R < 1$, suppose $a_p(0) = a(0)$, suppose $a_p \to a$, etc., almost everywhere, and suppose $e_p \to e$ and $f_p \to f$ strongly in $\mathfrak{L}_2$ on G_R.*

(i) *If $u \in \mathfrak{P}_2$, satisfies* (3.3), *and is a solution of* (3.1) *on G_R and if u_p is, for each p, that solution in $\mathfrak{P}_2$ of* $(3.1)_p$ *and* (3.3) *which $= u$ on S_R^-, then $u_p \to u$ strongly in $\mathfrak{P}_2$ on G_R.*

(ii) *If, for each p, u_p is a solution in $\mathfrak{P}_2$ of* $(3.1)_p$ *and* (3.3) *on each G_r with $r < R$ and if $u_p \to u$ strongly in $\mathfrak{L}_2$ on G_R, then $u \in \mathfrak{P}_2$ and is a solution of* (3.1) *and* (3.3) *on each G_r with $r < R$ and $u_p \to u$ weakly in $\mathfrak{P}_2$ on each such G_r.*

We come now to our principal theorem on differentiability:

THEOREM 3.4. *Snppose that u is any solution of* (3.1) *and* (3.3) *which $\in \mathfrak{P}_2$ on G_R.*

(i) *Suppose that a, b, b^*, and c satisfy the hypotheses of Theorem* 3.1(i), *suppose that $C_{11}R < 1$ and $R \leqq R_0$, and suppose e and $f \in \mathfrak{L}_{2\lambda}$, $\lambda = \rho - 1 + \mu$, $0 < \mu < 1$, an B_R. Then $\nabla u \in \mathfrak{L}_{2\lambda}$ and $u \in C_\mu{}^0$ on B_R.*

(ii) *Suppose also that b and $e \in C_\mu{}^0$ with b satisfying* (3.12) *on G_R and suppose that $C_{12} \cdot R < 1$ and $R \leqq R_0$. Then $u \in C_\mu{}^1$ on G_R.*

(iii) *If a, b, and $e \in C_1{}^0$, a, b, b^*, and c satisfy the hypotheses of* (i), *and $f \in \mathfrak{L}_2$ on B_R, then $\nabla u \in \mathfrak{P}_2$ on each G_r with $r < R$.*

(iv) *If a, b, and $e \in C_\mu{}^{k-2}$ and b^*, c, and $f \in C_\mu{}^{k-3}$, $0 < \mu < 1$, $k \geqq 3$, then $u \in C_\mu{}^{k-1}$ on each G_r with $r < R$.*

(v) *If a, b, b^*, c, e, and f are analytic on G_{R_0}, then u is analytic on G_R for each sufficiently small R.*

Proof. In (i) and (ii) our assumptions guarantee that $\| T \| < 1$. From the symmetry properties (3.6) and (3.7) for H and from Theorems

3.4 and 3.5 of part I, it follows that V belongs to whatever space is being considered. This is also true of W, using the theorems of Section 2.

The proofs of (iii) and (iv) are like those of (iv) and (v) of Theorem 4.4 of part I, except that γ is kept $\leqq n-1$ when applying the device of Lichtenstein. We conclude from this that all the $u_{x\gamma}$ for $\gamma \leqq n-1$ belong to $\mathfrak{P}_2$ and satisfy (3.3) on each G_r with $r < R$. Moreover, on spheres interior to G_R, the device may be applied with $\gamma = n$ showing that $u_{x^n} \varepsilon \mathfrak{P}_2$ on such spheres. But then a simple Green's theorem for $\mathfrak{P}_2$ functions allows us to replace equations (3.1) by the corresponding system of differential equations (almost everywhere). These may be solved for the $u_{x^n x^n}{}^i$ obtaining

$$(3.17)\qquad u_{x^n x^n}{}^i = \sum_{\alpha=1}^{n-1} A_{nj}{}^{i\alpha} u_{x^n x^\alpha}{}^j + \sum_{\alpha,\beta=1}^{n-1} B_j{}^{i\alpha\beta} u_{x^\alpha x^\beta}{}^j + C_j{}^{i\alpha} u_{x^\alpha}{}^j + D_j{}^i u^j + F^i$$

where in (v) all the coefficients are analytic if R_0 is small enough, in (iii) the A's and B's $\varepsilon\, C_1{}^0$, C and D are bounded and measurable and $F \varepsilon\, \mathfrak{L}_2$, and in (iv) the A's and B's $\varepsilon\, C_\mu{}^{k-2}$ and C, D, and $F \varepsilon\, C_\mu{}^{k-3}$. The result (iii) follows immediately. To obtain (iv), we first apply the device of Lichtenstein with $\gamma \leqq n-1$ on the whole of each G_r as many times as the coefficients allow. Then by applying the device with $\gamma = n$ on the interior of G_r and using (3.17) and its derivatives, we obtain the desired results.

In the analytic case (v), we conclude from (iv) that $u \varepsilon\, C^\infty$ on each G_r with $r < R$. If we choose R so small that $R \leqq R_0$ and $C_{13} R < 1$, the argument for (i) and (ii) shows that $u \varepsilon\, \mathfrak{B}^*{}_{0RK}{}^1$ from which we conclude using Lemma 2.3 that all the $'\nabla^p u$ and $'\nabla^p u_{x^n}$ are continuous in x and analytic in x'_n on each such G_r. Our previous regularity results show that u is analytic interior to such G_r. The $\mathfrak{B}^*{}_{0RK}{}^1$ bounds, together with repeated differentiations of (3.17) suffice to obtain the necessary bounds for the derivatives of u; or a sort of dominating function method like that in the Cauchy-Kowalewsky theorem may be used.

4. Manifolds with boundary. For an n-dimensional Riemannian manifold $\mathfrak{M}$ with boundary $\mathfrak{B}$ of class $C_\mu{}^k$ $(0 \leqq \mu \leqq 1)$, (C^∞, analytic) we adopt the standard definition: each point of $\mathfrak{M}(U\mathfrak{B})$ is contained in some set $\mathfrak{N}$ open on $\mathfrak{M}$ which is either the homeomorphic image of the unit ball or of the part of it where $x^n \leqq 0$ in which latter case, the points where $x^n = 0$ correspond to $\mathfrak{N} \cap \mathfrak{B}$; any two overlapping coordinate systems are related by a transformation of class $C_\mu{}^k$ (C^∞, or analytic). An admissible coordinate system for such a manifold will be any homeomorphism of a Lipschitzian

domain in G in E^n onto a set $\mathfrak{N}$ open on $\mathfrak{M}$ which is related to the "preferred," coordinate systems above by a transformation of class $C_\mu{}^k$ (C^∞, or analytic); if $\mathfrak{N} \cap \mathfrak{B}$ is not empty, G must lie in the half-plane $x^n < 0$ and the part of ∂G on $x^n = 0$ must correspond to $\mathfrak{N} \cap \mathfrak{B}$. In any admissible coordinate system, the g_{ij} are of class $C_\mu{}^{k-1}$.

As in part I, we shall assume that $\mathfrak{M}$ is at least of class $C_1{}^1$, in which case the g_{ij} are merely Lipschitzian. We shall be concerned with exterior differential forms on $\mathfrak{M}$. Since we have not required (and shall not) $\mathfrak{M}$ to be orientable, we shall consider both *even* and *odd* forms (see [2], § 3) on $\mathfrak{M}$. The law of transformation of the components under coordinate transformations is

$$(4.1)\qquad \begin{gathered} {}'\omega_{i_1\cdots i_r}(x') = \epsilon \sum_{j_1<\cdots<j_r} \omega_{j_1\cdots j_r}[x('x)] \frac{\partial(x^{j_1},\cdots,x^{j_r})}{\partial('x^{i_1},\cdots,'x^{i_r})}, \\ \epsilon = \begin{cases} +1, & \text{for even forms,} \\ J/|J|, & \text{for odd forms.} \end{cases} \qquad J = \frac{\partial(x^1,\cdots,x^n)}{\partial('x^1,\cdots,'x^n)}. \end{gathered}$$

The differentiability class of a form ω is that of its components in all coordinate systems; on a manifold of class $C_\mu{}^k$, it is clear that no form can be of class $\geqq C_\mu{}^{k-1}$ (although in particular coordinate systems the components might have higher class). Forms of class $\mathfrak{L}_2$ are defined as usual and the $\mathfrak{L}_2$ inner product of two (both even or both odd) forms ω and η and the norm of ω will be denoted by (ω, η) and $|\omega|$, respectively. Forms of class $\mathfrak{P}_2$ were defined in part I for manifolds without boundary; it is clear that the definition there given carries over to the present case along with those of the $\mathfrak{P}_2$ inner product and norm $((\omega,\eta))$ and $\|\omega\|$, depending on a finite number $\mathfrak{N}$ of admissible coordinate systems covering $\mathfrak{M}$. The differential operators d and δ and the Dirichlet integral are defined as usual (see part I, § 5). The dual operator * is defined as usual (see, for instance [3], p. 129 or [2]).

There are parallel theories for even and odd forms. From now on we assume that all forms are of some one kind.

We have immediately the theorem:

THEOREM 4.1. *The spaces $\mathfrak{L}_2$ and $\mathfrak{P}_2$ of even or odd forms are Hilbert spaces with the norms above. The operators d and δ are bounded operators from $\mathfrak{P}_{2e}$ to $\mathfrak{L}_{2e}$ and from $\mathfrak{P}_{2o}$ to $\mathfrak{L}_{2o}$ and the Dirichlet integral $D(\omega)$ is lower-semicontinuous with respect to weak convergence in either $\mathfrak{P}_{2e}$ or $\mathfrak{P}_{2o}$. Finally if $\omega_p \to \omega$ weakly in $\mathfrak{P}_{2e}$ (or $\mathfrak{P}_{2o}$) then $\omega_p \to \omega$ strongly in $\mathfrak{L}_{2e}$ (or $\mathfrak{L}_{2o}$).*

We shall be concerned with the boundary values of forms. We begin with the following definition:

Definition 4.1. By the *boundary values* of a form on $\mathfrak{M}(U\mathfrak{B})$, we merely mean that form restricted to $\mathfrak{B}$; that is the boundary value $b\omega$ of ω is given by

$$b\omega = \sum_{i_1 < \cdots < i_r \leqq n} \omega_{i_1 \cdots i_r}(0, x'_n)\, dx^{i_1} \cdots dx^{i_r} \tag{4.2}$$

whenever $\omega_{i_1 \cdots i_r}(x)$ are the components of ω in an admissible boundary coordinate system. The boundary value of ω is said to be *of class* $\mathfrak{L}_2$ *or* $\mathfrak{P}_2$ *along* $\mathfrak{M}$, if $\mathfrak{M}$ is of class $C_1{}^1$, or of class $C_\nu{}^l \leqq C_\mu{}^{k-1}$ ($C_\mu{}^k$ for 0-forms) if $\mathfrak{M}$ is of class $C_\mu{}^k$, or C^∞, or analytic, if and only if its components under any admissible boundary coordinate system have the indicated class as functions of x'_n. If ω is of class $\mathfrak{P}_2$ on $\mathfrak{M}$, it is understood that the components of ω are to be replaced by the corresponding functions of Theorem 2.7 of part I as in Theorems 2.9, etc., of part I.

Before proceeding further, we prove the following lemma:

LEMMA 4.1. (a) *If* $\omega \,\varepsilon\, \mathfrak{P}_2$ *on* $\mathfrak{M}$, *its boundary value* $\varepsilon\, \mathfrak{L}_2$ *on* $\mathfrak{B}$.

(b) *If the boundary value of some form* ω_0 *is of class* $\mathfrak{P}_2$ *along the boundary* $\mathfrak{B}$ *of a manifold* $\mathfrak{M}$ *of class* $C_1{}^1$, *then there is a form* $\omega \,\varepsilon\, \mathfrak{P}_2$ *on* $\mathfrak{M}$ *with the same boundary value almost everywhere on* $\mathfrak{M}$. *If, also* $d\omega_0(\delta\omega_0) \,\varepsilon\, \mathfrak{P}_2$ *along* $\mathfrak{B}$, *we may choose* ω *so that* $d\omega(\delta\omega) \,\varepsilon\, \mathfrak{P}_2$ *on* $\mathfrak{M}$.

(c) *If* $\mathfrak{M}$ *is of class* $C_\mu{}^k$ (C^∞) *and the boundary value of* ω_0 *is of class* $C_\mu{}^{k-1}$ ($C_\mu{}^k$ *for* 0 *forms*) (C^∞) *along* $\mathfrak{B}$, *then there is a form* ω *of class* $C_\mu{}^{k-1}$ ($C_\mu{}^k$ *for* 0 *forms*) (C^∞) *on* $\mathfrak{M}$ *with the same boundary value. If we also assume* $d\omega(\delta\omega)$ *of class* $C_\mu{}^{k-1}$ (*on* $\mathfrak{M}$ *of class* $C_\mu{}^k$) *we may choose* ω *so that* $d\omega(\delta\omega) \,\varepsilon\, C_\mu{}^{k-1}$ *on* $\mathfrak{M}$.

(d) *If* $\mathfrak{M}$ *is analytic and the boundary value of* ω_0 *is analytic along* $\mathfrak{B}$, *then* ω *may be chosen to be of class* C^∞ *on* $\mathfrak{M}$ *and analytic in some neighborhood of any given boundary point.*

Proofs. (a) follows from Theorem 2.9 of part I. We prove (b) and (c) simultaneously as follows: We may cover B with the ranges $\mathfrak{N}_1, \cdots, \mathfrak{N}_Q$ of admissible boundary coordinate systems whose domains are all the part $x^n \leqq 0$ of the unit ball $B(0;1)$, and we may find a partition of unity $\phi_1, \cdots, \phi_S$, each of whose functions is of class $C_1{}^1(C_\mu{}^k)$, and such that each function whose support intersects $\mathfrak{B}$ has support in some one $\mathfrak{N}_q$. Since the

sum of all those $\phi_s(P)$ whose support intersects $\mathfrak{B}$ is 1 on $\mathfrak{B}$, we may define $\omega=\omega_1+\cdots+\omega_S$ where each $\omega_s=0$ unless the support of ϕ_s intersects $\mathfrak{B}$ in which case we define $\omega_s=0$ outside $\mathfrak{R}_q$ and define ω_s in $\mathfrak{R}_q$ by

$$(4.2)\qquad \omega^{(q)}{}_{si_1\cdots i_r}(x^n,x'_n)=\omega^{(q)}{}_{0i_1\cdots i_r}(0,x'_n)\cdot\phi_s{}^{(q)}(x^n,x'_n).$$

The differentiability results, including those about $d\omega$ follows from (4.2); to get the ones about $\delta\omega$, begin by taking duals. In the analytic case we make sure that one of the $\mathfrak{R}_q$ is a boundary coordinate system about the given point P and that one $\phi_s\equiv 1$ in some neighborhood of P.

Definition 4.2. Suppose ω is defined on $\mathfrak{M}$. Then we define the *tangential* and *normal parts* $t\omega$ and $n\omega$ of its boundary value by the condition that

$$(4.3)\qquad t\omega=\sum_{i_1<\cdots<i_r<n}\omega_{i_1\cdots i_r}dx^{i_1}\cdots dx^{i_r},\quad n\omega=b\omega-t\omega,\ \text{on}\ x^n=0$$

in any admissible boundary coordinate system in which

$$(4.4)\qquad \omega=\sum_{i_1<\cdots<i_r\leqq n}\omega_{i_1\cdots i_r}dx^{i_1}\cdots dx^{i_r}\ \text{on}\ x^n=0.$$

Since $\partial x^n/\partial' x^i=0$ on $'x^n=0$ in the relation between two overlapping admissible boundary coordinate systems, it follows that $t\omega$ and hence $n\omega$ is invariantly defined on $\mathfrak{B}$.

The following lemma is important:

LEMMA 4.2. (a) *If $\omega_p\rightarrow\omega$ weakly in $\mathfrak{P}_2$ on $\mathfrak{M}$, then $b\omega_p\rightarrow b\omega$, $t\omega_p\rightarrow t\omega$, and $n\omega_p\rightarrow n\omega$ strongly in $\mathfrak{L}_2$ on $\mathfrak{B}$.*

(b) *If $\omega\,\varepsilon\,\mathfrak{P}_2$ on $\mathfrak{M}$, then $t(*\omega)=*(n\omega)$ and $n(*\omega)=*(t\omega)$ and $*(b\omega)=b(*\omega)$ on $\mathfrak{B}$. Here $*(n\omega)$ and $*(t\omega)$ may be found by first extending $n\omega$ and $t\omega$ to $\mathfrak{M}$ in any way, taking the duals, and then finding the boundary values; the result is independent of the way $n\omega$ and $t\omega$ are extended.*

Proofs. (a) follows from Theorems 2.11 and 2.12 of part I. From our strong convergence theorems, it suffices to prove (b) for Lipschitz forms. Let ω be such a form and let P be any point of $\mathfrak{B}$. There exists an admissible boundary coordinate system with domain B_R which carries the origin into P with $g_{ij}(0)=\delta_{ij}$. In that coordinate system, we have (see formula 1.10) of [3], for instance)

$$(4.5)\qquad (*\omega)_{i_1\cdots i_{n-r}}=e_{j_1\cdots j_r i_1\cdots i_{n-r}}\omega_{j_1\cdots j_r}\qquad (j\ \text{not summed})\ \text{at}\ x=0,$$

the j's being those positive integers $\leqq n$ which are not among the i's. The results follow from (4.5) and the definitions.

If $\mathfrak{M}$ is of class C''' and ϕ and ψ are forms of class C'' of the same kind and of degrees p and $(p+1)$, respectively, the following formula is derived in [3], § 2, from Stokes formula:

$$(d\phi, \psi) - (\phi, \delta\psi) = \int_{\mathfrak{B}} \phi \wedge * \psi. \tag{4.6}$$

From this, it follows that if $\mathfrak{M}$ is of class C'''', ω is of class C''' and ζ is of class C'', then

$$(d\omega, d\zeta) + (\delta\omega, \delta\zeta) = (\Delta\omega, \zeta) + \int_{\mathfrak{B}} (\zeta \wedge * d\omega - \delta\omega \wedge * \zeta), \tag{4.7}$$
$$\Delta = \delta d + d\delta,$$

Δ being the Laplacian operator as used in [2] and [3] (see [3], p. 130). In Euclidean space with Cartesian coordinates, this is the negative of the ordinary Laplacian. In this connection, we note that if $\mathfrak{M} = G_R$, the part of the sphere $|x| < R$ for which $x^n < 0$, then

$$\begin{aligned} D_0(\omega) = 2 \int_{\sigma_R} (-1)^{n-r} \sum_{i_1 < \cdots i_r < n} \sum_{k=1}^{r} (-1)^k [&\omega_{i_1 \cdots i_r} \omega_{i_1 \cdots i'_k \cdots i_r n x^{i_k}} \\ &- \omega_{i_1 \cdots i_r x^{i_k}} \omega_{i_1 \cdots i'_k \cdots i_r n}] dx_1 \cdots dx_{n-1} \\ &+ \int_{G_R} \sum_{i_1 < \cdots < i_r} \sum_{\alpha=1}^{n} (\omega_{i_1 \cdots i_r x^\alpha})^2 dx \end{aligned} \tag{4.8}$$

for any form ω of class C'' which is zero on and near the spherical surface of G_R, $D_0(\omega)$ being the Dirichlet integral $D(\omega)$ referred to Cartesian coordinates and σ_R being the part of G_R for which $x^n = 0$.

LEMMA 4.3. (a) *If ω (considered as a set of functions) $\varepsilon\, \mathfrak{P}_2$ on G_R, ω is zero on and near the spherical part of the surface of G_R, and if either $t\omega = 0$ or $n\omega = 0$, there exists a sequence ω_p of forms of class C^k for any desired k, on G_R which converge strongly in $\mathfrak{P}_2$ on G_R to ω and such that $t\omega_p = 0$ or $n\omega_p = 0$ (respectively) for each p.*

(b) *If ω satisfies the hypotheses of* (a), *then*

$$D_0(\omega) = \int_{G_R} \sum_{(i)\alpha} \omega_{(i)x^\alpha})^2 \, dx. \tag{4.9}$$

Proof. Clearly (b) follows from (a) and (4.8). Also, since the condition $t\omega = 0$ is just the same as saying that the ω's with $i_r < n$ vanish on σ_R and $n\omega = 0$ is the same as saying that those with $i_r = n$ vanish on σ_R, part (a) is just reduced to proving the theorem for functions. If the

function ω is not required to be zero on σ_R, we extend ω to the whole of B_R by $\omega(-x^n, x'_n) = \omega(x^n, x'_n)$ and then note that the first spherical h-averages of ω are of class C' and have support $\subset B_R$ if h is small enough; these tend strongly in $\mathfrak{P}_2$ to ω on B_R. We may then repeat the process $k-1$ more times. If ω is zero on σ_R, we begin by extending ω to B_R by $\omega(-x^n, x'_n) = -\omega(x^n, x'_n)$ and then proceeding as above; we note that each of the successive averages vanishes on σ_R.

This lemma and equation (4.6) for smooth forms and manifolds allows us to prove the following important facts:

LEMMA 4.4. *Suppose α and β are any forms $\varepsilon\, \mathfrak{P}_2$ which are of the same kind and of proper degrees. Then $(\delta\alpha, d\beta) = 0$ and $(\alpha, d\beta) = (\delta\alpha, \beta)$, if either $n\alpha$ or $t\beta = 0$.*

Proof. We may select a finite covering of $\mathfrak{M}$ by coordinate neighborhoods $\mathfrak{N}_1, \cdots, \mathfrak{N}_Q$ covering $\mathfrak{M}$, the coordinate system corresponding to any boundary neighborhood being an admissible boundary coordinate system. It is clear that there is a number $R_0 > 0$ such that any geodesic sphere on $\mathfrak{M}$ of radius R_0 is contained in some one $\mathfrak{N}_q$. A finite number of new neighborhoods, each part of a sphere of radius $R_0/3$, also covers $\mathfrak{M}$. If we choose a partition of unity $\phi_1 + \cdots + \phi_S \equiv 1$, each function of which is Lipschitz and has support in one of these small neighborhoods we see that

$$(\delta\alpha, d\beta) = \sum_{s,t} (\delta\alpha_s, d\beta_t), \qquad \alpha_s = \phi_s\alpha, \beta_t = \phi_t\beta$$

where the sum is extended over all ordered pairs (s, t) such that the supports of α_s and β_t intersect. But for any such pair, the union of the supports is included in some one $\mathfrak{N}_q$ so our problem is reduced to that case. But for inner neighborhoods the theorem has been proved in part I, Lemma 7.1. But the same proof extends to boundary neighborhoods using Lemma 4.3.

LEMMA 4.5. (Gaffney [5]). *With each point P of $\mathfrak{M}$ and each $\epsilon > 0$ is associated an admissible coordinate system $\mathfrak{C}$ with domain G and range $\mathfrak{U}$ and a constant l such that*

$$D(\omega) \geqq (1-\epsilon) \int_G \sum_{i,\alpha} (\omega_{ix^\alpha})^2 \, dx - l(\omega, \omega)$$

for any form $\omega\, \varepsilon\, \mathfrak{P}_2$ with support on $\mathfrak{U}$ and either $t\omega = 0$ or $n\omega = 0$ on $\mathfrak{B}$.

Proof. This has been proved for interior points in part I, Lemma 5.1. If P is a boundary point, it is clear that we may choose an admissible

boundary coordinate system with domain $G_R \cup \sigma_R$ which carries the origin into P and $G_R \cup \sigma_R$ into a neighborhood of P in which $g_{ij}(0) = \delta_{ij}$. Then, exactly as in the proof for interior points, we conclude that we may choose R so small that

$$D(\omega) \geqq D_0(\omega) - \epsilon \int_{G_R} \sum_{i,\alpha} (\omega_{ix^\alpha})^2 \, dx - l(\omega, \omega)$$

for some l and all $\omega \in \mathfrak{P}_2$ with support in $\mathfrak{N}$, $D_0(\omega)$ having its significance in Lemma 4.3. The result follows from that lemma.

The following theorem follows from the lemma above in exactly the same way as in the case of part I, Theorem 5.4.

THEOREM 4.2. *For each finite system $\mathfrak{N}$ of admissible coordinate systems whose ranges cover $\mathfrak{M}$, there are constants k and l such that*

$$D(\omega) \geqq k \| \omega \|^2 - l(\omega, \omega) \qquad (k > 0)$$

for any form in $\mathfrak{P}_2$ with either $t\omega = 0$ or $n\omega = 0$ on $\mathfrak{B}$, the norm being that corresponding to $\mathfrak{N}$.

5. Potentials; the decomposition theorem. In this and the next section, we shall assume that all of our forms are of the same kind, completely parallel theories being obtained for each kind.

Definition 5.1. We define the *closed linear manifolds $\mathfrak{P}_2^+$ and $\mathfrak{P}_2^-$* (see Theorem 4.2) of $\mathfrak{P}_2$ as the totality of forms in $\mathfrak{P}_2$ for which $n\omega = 0$ and $t\omega = 0$ on $\mathfrak{B}$, respectively.

Just as in part I, Section 6, we obtain the following result:

LEMMA 5.1. *Let M be any closed linear manifold of $\mathfrak{L}_2$ such that $M \cap \mathfrak{P}_2^+(\mathfrak{P}_2^-)$ is not empty. Then there exists a form ω in $M \cap \mathfrak{P}_2^+(\mathfrak{P}_2^-)$ which minimizes $D(\omega)$ among all such forms with $(\omega, \omega) = 1$.*

THEOREM 5.1. *The manifold $\mathfrak{H}^+(\mathfrak{H}^-)$ of harmonic fields in $\mathfrak{P}_2^+(\mathfrak{P}_2^-)$ is finite dimensional.*

THEOREM 5.2. *If $\omega \in \mathfrak{P}_2^+(\mathfrak{P}_2^-)$ and is $\mathfrak{L}_2$-orthogonal to $\mathfrak{H}^+(\mathfrak{H}^-)$, then there are positive constants λ^+ and λ^- for each $\mathfrak{N}$ such that*

$$D(\omega) \geqq \lambda^+ \| \omega \|^2 (\lambda^- \| \omega \|^2).$$

THEOREM 5.3. *If $\eta \in \mathfrak{L}_2$ and is $\mathfrak{L}_2$-orthogonal to $\mathfrak{H}^+(\mathfrak{H}^-)$, there is a unique form $\Omega^+(\Omega^-)$ in $\mathfrak{P}_2^+ \perp \mathfrak{H}^+ (\mathfrak{P}_2^- \perp \mathfrak{H}^-)$ such that*

$$(d\Omega^+, d\zeta) + (\delta\Omega^+, \delta\zeta) = (\eta, \zeta), \zeta \in \mathfrak{P}_2^+(\mathfrak{P}_2^-). \tag{5.1}$$

Definition 5.2. The functions Ω^+ and Ω^- are called the plus-potential and minus-potential of η, respectively.

The defining equations (5.1) for the potentials are a special case of the more general equations

$$(5.2) \qquad (d\omega - \phi, d\zeta) + (\delta\omega - \psi, \delta\zeta) - (\eta, \zeta) = 0, \; \zeta \,\varepsilon\, \mathfrak{P}_2^+ \text{ or } \mathfrak{P}_2^-$$

which were discussed in Section 7, part I. The differentiability results for such equations on the interior of $\mathfrak{M}$ follow from the discussion there given. But now, suppose we select a point P on $\mathfrak{B}$ and choose an admissible boundary coordinate system with domain G_R and range a boundary neighborhood $\mathfrak{U}$ of P such that $g_{ij}(0) = \delta_{ij}$. In such a system the conditions $n\omega = n\zeta = 0$ for $\mathfrak{P}_2^+$ and $t\omega = t\zeta = 0$ for $\mathfrak{P}_2^-$ correspond under a proper ordering of the sets (i) and (j) to the equations (3.3). Accordingly as in Section 7, part I, we see that if the support of ζ is confined to $\mathfrak{U}$, the system (5.2) reduces to the system (3.1) and (3.3) discussed in Section 3. Since the theorems of Section 3 parallel exactly those of part I, Section 4, we may conclude that the differentiability results for the plus and minus potentials stated near the beginning of Section 7, part I, hold right up to the boundary. We now extend these results as in Section 7, part I, and summarize as follows:

THEOREM 5.4. *Suppose* $\omega \,\varepsilon\, \mathfrak{L}_2 \ominus \mathfrak{H}^+ (\mathfrak{L}_2 \ominus \mathfrak{H}^-)$ *and* Ω *is its plus (minus)-potential.*

(i) *If* $\mathfrak{M}$ *is of class* C_1^1, *then* Ω, $d\Omega$, *and* $\delta\Omega$ *are in* $\mathfrak{P}_2^+(\mathfrak{P}_2^-)$.

(ii) *If* $\mathfrak{M}$ *is of class* C_1^1 *and* ω *is in* $\mathfrak{L}_{2\lambda}$ *with* $\lambda = \rho - 1 + \mu (\rho = n/2)$, *then* Ω, $d\Omega$, *and* $\delta\Omega$ *are in* $\mathfrak{P}_{2\lambda}^+(\mathfrak{P}_{2\lambda}^-)$ *and* C_μ^0 *if* $0 < \mu < 1$.

(iii) *If* $\mathfrak{M}$ *is of class* C_μ^k *and* $\omega \,\varepsilon\, C_\mu^{k-2}$ $(k \geqq 2, 0 < \mu < 1)$, *then* Ω, $d\Omega$, *and* $\delta\Omega \,\varepsilon\, C_\mu^{k-1}$. *If* $k \geqq 3$ *and* $\omega \,\varepsilon\, C_\mu^{k-3}$, *then* $\Omega \,\varepsilon\, C_\mu^{k-1}$.

(iv) *If* $\mathfrak{M}$ *and* ω *are of class* C^∞ *or analytic, then so is* Ω.

(v) *If* Ω *and* ω *are 0-forms, then* Ω *has an additional degree of differentiability in all cases above except the second half of* (iii).

In all cases, if we set $\alpha = d\Omega$ *and* $\beta = \delta\Omega$,

$$(5.3) \qquad \delta\alpha + d\beta = \delta(d\Omega) + d(\delta\Omega) = \omega, \qquad d\alpha = \delta\beta = 0;$$

$$(5.4) \qquad (d\alpha, d\zeta) + (\delta\alpha - \omega, \delta\zeta) = (d\beta - \omega, d\zeta) + (\delta\beta, \delta\zeta) = 0, \; \zeta \,\varepsilon\, \mathfrak{P}_2^+(\mathfrak{P}_2^-).$$

Proof. The results for Ω follows directly from the discussion above and

Section 3. The proof of the results for $d\Omega$ and $\delta\Omega$ is like that of Theorem 7.1 of part I where it is already done for the interior of $\mathfrak{M}$. We choose a boundary point P and an admissible boundary coordinate system of the type described in the preceding paragraph and approximate (if necessary) to ω and the g_{ij} by smooth functions. For each of the approximating functions Ω, we see from formula (4.7) that $\Delta\Omega = \omega$ and

$$(5.5) \qquad t^*d\Omega = 0 \text{ if } \Omega \,\varepsilon\, \mathfrak{P}_2^+ \text{ and } t\delta\Omega = 0 \text{ if } \Omega \,\varepsilon\, \mathfrak{P}_2^-$$

since the integral over $\mathfrak{B}$ depends only on the tangential parts of each factor in each term and $t\zeta$ is arbitrary if $\zeta \,\varepsilon\, \mathfrak{P}_2^+$ and $n\zeta$ or $t^*\zeta$ is arbitrary if $\zeta \,\varepsilon\, \mathfrak{P}_2^-$. From Stokes' theorem we see that $td\phi = 0$ whenever $t\phi = 0$ and ϕ and $\mathfrak{M}$ are differentiable. From Lemma 4.2(b) and the formula for δ, it follows from this that

$$(5.6) \qquad n\phi = 0 \rightarrow t^*\phi = 0 \rightarrow td^*\phi = 0 \rightarrow n^*d^*\phi = 0 \rightarrow n\delta\phi = 0.$$

Hence, from this and (5.5), we see that both α and $\beta \,\varepsilon\, \mathfrak{P}_2^+(\mathfrak{P}_2^-)$ if ω and $\Omega \,\varepsilon\, \mathfrak{P}_2^+(\mathfrak{P}_2^-)$ at each stage of the approximation, that α and β satisfy (5.3), and hence (5.4) using Lemma 4.4. The approximation may then be carried through as before on each G_r with $r < R$. Since a finite number of the smaller boundary neighborhoods cover $\mathfrak{B}$, the results (5.4) for all ζ in $\mathfrak{P}_2^+(\mathfrak{P}_2^-)$ follow and the differentiability of α and β now follow from Section 3.

Remark. Except in the case of zero forms Ω, the individual derivatives of the individual components of Ω *do not,* in general, have the same differentiability properties as do $d\Omega$ and $\delta\Omega$ (the coordinate transformations will not allow it).

The following two theorems are useful and important:

THEOREM 5.5. *Suppose $\eta \,\varepsilon\, \mathfrak{P}_2$, H^+ and H^- are its projections in $\mathfrak{H}^+$ and $\mathfrak{H}^-$ and Ω^+ and Ω^- are the plus and minus potentials of $\eta - H^+$ and $\eta - H^-$, respectively, and $\alpha^\pm = d\Omega^\pm$, $\beta^\pm = \delta\Omega^\pm$. Then*

(i) *α^+ is the plus potential of $d\eta$ and β^- is the minus potential of $\delta\eta$ and $d\eta \,\varepsilon\, \mathfrak{L}_2 \ominus \mathfrak{H}^+$ and $\delta\eta \,\varepsilon\, \mathfrak{L}_2 \ominus \mathfrak{H}^-$.*

(ii) *If $\eta \,\varepsilon\, \mathfrak{P}_2^+$, then β^+ is the plus potential $\delta\eta \,\varepsilon\, \mathfrak{L}_2 \ominus \mathfrak{H}^+$.*

(iii) *If $\eta \,\varepsilon\, \mathfrak{P}_2^-$, then α^- is the minus potential of $d\eta \,\varepsilon\, \mathfrak{L}_2 \ominus \mathfrak{H}^-$.*

Proof. These results follow from Theorem 5.4, equation (5.4), and Lemma 4.4.

THEOREM 5.6. (i) *If* $\eta^+ \varepsilon \mathfrak{P}_2^+$ *and* $\eta^- \varepsilon \mathfrak{P}_2^-$, *there are unique forms* α *and* β, *where* $\alpha \varepsilon \mathfrak{P}_2^+$ *and is* $\mathfrak{L}_2$*-orthogonal to* $\mathfrak{H}^+$ *and* $\beta \varepsilon \mathfrak{P}_2^-$ *and is* $\mathfrak{L}_2$*-orthogonal to* $\mathfrak{H}^-$ *such that* $\delta\alpha = \delta\eta^+$, $d\alpha = 0$, $d\beta = d\eta^-$, $\delta\beta = 0$.

(ii) *If* $\eta \varepsilon \mathfrak{P}_2$, *there are unique forms* $\gamma \varepsilon \mathfrak{P}_2^+ \cap (\mathfrak{L}_2 - \mathfrak{H}^+)$ *and* ϵ *in* $\mathfrak{P}_2^- \cap (\mathfrak{L}_2 - \mathfrak{H}^-)$ *such that* $d\gamma = d\eta$, $\delta\gamma = 0$, $\delta\epsilon = \delta\eta$, $d\epsilon = 0$.

Proof. The uniqueness is evident. To prove (i), let Ω^+ and Ω^- be the respective plus and minus potentials of $\eta^+ - H^+$ and $\eta^- - H^-$ and let $\Gamma = \delta\Omega^+$ and $E = d\Omega^-$. From Theorem 5.5, we see that Γ is the plus potential of $\delta\eta^+$ and E is the minus potential of $d\eta^-$. Then, from Theorem 5.4 we conclude that $\alpha = d\Gamma$ and $\beta = \delta E$ have the desired properties. To prove (ii), let Ω^+ and Ω^- be the respective plus and minus potentials of $\eta - H^+$ and $\eta - H^-$ and let

$$A = d\Omega^+, \quad B = \delta\Omega^-, \quad \gamma = \delta A, \quad \epsilon = dB$$

and (ii) follows from Theorem 5.5(i).

Definition 5.3. We define the *linear sets* $\mathfrak{C}$ *and* $\mathfrak{D}$ as the sets of all forms of the form $\delta\alpha$ and $d\beta$, where $\alpha \varepsilon \mathfrak{P}_2^+$ and $\beta \varepsilon \mathfrak{P}_2^-$, respectively.

We now can prove an analog for the Kodaira decomposition theorem [6] for manifolds with boundary.

THEOREM 5.7. *The sets* $\mathfrak{C}$ *and* $\mathfrak{D}$ *and the set* $\mathfrak{H}$ *of all harmonic fields in* $\mathfrak{L}_2$ *on* $\mathfrak{M}$ *are closed linear manifolds in* $\mathfrak{L}_2$ *and*

$$\mathfrak{L}_2 = \mathfrak{C} \oplus \mathfrak{D} \oplus \mathfrak{H}. \tag{5.7}$$

Moreover, if $\omega \varepsilon \mathfrak{P}_2$, *its* $\mathfrak{L}_2$ *projections* γ, ϵ, *and* H *on* $\mathfrak{C}$, $\mathfrak{D}$, *and* $\mathfrak{H}$ *belong to* $\mathfrak{P}_2^+$, $\mathfrak{P}_2^-$, *and* $\mathfrak{P}_2$, *respectively, and* $\delta\gamma = d\epsilon = 0$.

Proof. That $\mathfrak{C}$ and $\mathfrak{D}$ are closed linear manifolds follows immediately from Theorems 5.6 and 5.2 and that $\mathfrak{H}$ is also follows from the theorems of part I, Sections 4 and 7. Using Lemma 4.4, we see that $\mathfrak{C}$ and $\mathfrak{D}$ are orthogonal and that $\mathfrak{C}$ and $\mathfrak{D}$ are both orthogonal to $\mathfrak{H} \cap \mathfrak{P}_2$ and, in fact, if $H \varepsilon \mathfrak{P}_2 \cap (\mathfrak{L}_2 \ominus \mathfrak{C} \ominus \mathfrak{D})$, then $H \varepsilon \mathfrak{H}$ ($\mathfrak{P}_2^+$ and $\mathfrak{P}_2^-$ are both everywhere dense in $\mathfrak{L}_2$).

Now, suppose $\eta \varepsilon \mathfrak{P}_2$ and let γ and ϵ be its projections on $\mathfrak{C}$ and $\mathfrak{D}$, respectively. Using Theorem 5.6 we conclude the existence of unique forms α and β in $\mathfrak{P}_2^+ \cap (\mathfrak{L}_2 \ominus \mathfrak{H}^+)$ and $\mathfrak{P}_2^- \cap (\mathfrak{L}_2 \ominus \mathfrak{H}^-)$ respectively, such that

$$\delta\alpha = \gamma, \quad d\alpha = 0, \quad \delta\beta = 0, \quad d\beta = \epsilon. \tag{5.8}$$

Since γ and ϵ are the projections of η on $\mathfrak{C}$ and $\mathfrak{D}$, we see from (5.8) that α and β satisfy

$$(5.9)\quad \begin{aligned}(d\alpha, d\zeta^+) + (\delta\alpha, \delta\zeta^+) - (\eta, \delta\zeta^+) = (d\alpha, d\zeta^+) + (\delta\alpha, \delta\zeta^+) - (d\eta, \zeta^+) = 0\\ (d\beta, d\zeta^-) + (\delta\beta, \delta\zeta^-) - (\eta, d\zeta^-) = (d\beta, d\zeta^-) + (\delta\beta, \delta\zeta^-) - (\delta\eta, \zeta^-) = 0\end{aligned}$$

for all ζ^+ in $\mathfrak{P}_2^+$ and ζ_2^- in $\mathfrak{P}_2^-$. Thus α is the plus potential of $d\eta$ and β is the minus potential of $\delta\eta$. The results follow from Theorems 5.3 and 5.5.

The following theorem contains further information concerning the decomposition (5.7):

THEOREM 5.8. *Suppose $\omega \varepsilon \mathfrak{L}_2$ and γ, ϵ, and H are its projections on $\mathfrak{C}$, $\mathfrak{D}$, and $\mathfrak{H}$, respectively, and suppose Ω^+ and Ω^- are the plus and minus potentials of $\omega - H$, respectively. Then*

$$(5.10)\qquad \gamma = \delta\alpha, \quad \epsilon = d\beta, \quad \alpha = d\Omega^+, \quad \beta = \delta\Omega^-, \quad d\alpha = \delta\beta = 0.$$

If $\omega \varepsilon \mathfrak{P}_2$, then α and β are the plus and minus potentials of $d\omega$ and $\delta\omega$, respectively. We have the following differentiability results on the closure of $\mathfrak{M}$:

(i) *If $\mathfrak{M}$ is of class C_1^1 and $\omega \varepsilon \mathfrak{L}_{2\lambda}$, then γ, ϵ, and $H \varepsilon \mathfrak{L}_{2\lambda}$ and α, β, Ω^+, and $\Omega^- \varepsilon \mathfrak{P}_{2\lambda}$; if also $\omega \varepsilon \mathfrak{P}_{2\lambda}$, then γ, ϵ, and $H \varepsilon \mathfrak{P}_{2\lambda}$ with γ, δ and H in $C_\mu{}^0$ in case $\lambda = \rho - 1 + \mu$, $0 < \mu < 1$.*

(ii) *If $\mathfrak{M}$ is of class $C_\mu{}^k$ with $k \geqq 2$* and $0 < \mu < 1$ *and if $\omega \varepsilon C_\mu{}^{k-2}$, then H, γ, and $\epsilon \varepsilon C_\mu{}^{k-2}$ and α, β, Ω^+, and $\Omega^- \varepsilon C_\mu{}^{k-1}$; if, also, $\omega \varepsilon C_\mu{}^{k-1}$, then H, γ, and $\epsilon \varepsilon C_\mu{}^{k-1}$.*

(iii) *If $\mathfrak{M}$ and ω are C^∞ or analytic, so are α, β, γ, ϵ, Ω^+, and Ω^-.*

(iv) *In the case of zero forms, H is a constant, and $\epsilon = 0$.*

If $\omega \varepsilon \mathfrak{P}_2$ with $d\omega = 0$ or if $\omega \varepsilon \mathfrak{L}_2$ and $\omega = d\eta$ where $\eta \varepsilon \mathfrak{P}_2$, then $\gamma = 0$; if $\omega \varepsilon \mathfrak{P}_2$ and $\delta\omega = 0$ or if $\omega \varepsilon \mathfrak{L}_2$ and $\omega = \delta\eta$ where $\eta \varepsilon \mathfrak{P}_2$ then $\epsilon = 0$.

Proof. Suppose, first, that $\omega \varepsilon \mathfrak{P}_2$. If we then define α, β, γ, ϵ, Ω^+, and Ω^- by (5.10), the results follow from (5.8) and (5.9). In case ω is merely in $\mathfrak{L}_2$, we use the left sides of (5.9) and approximate, using Theorems 3.3 and 5.2. The regularity results and the last statement follow from the facts that α and β are the respective potentials of $d\omega$ and $\delta\omega$, since $dH = \delta H = 0$, in case $\omega \varepsilon \mathfrak{P}_2$. The last results for ω merely in $\mathfrak{L}_2$ follow from Lemma 4.4.

We may now prove a slightly strengthened form of an inequality due to Friedrichs [4]:

THEOREM 5.9. *There is a* $\lambda > 0$ *such that if* $\omega \varepsilon \mathfrak{P}_2 \perp \mathfrak{H}$, *then*

$$D(\omega) \geqq \lambda \| \omega \|^2.$$

Proof. For if $\omega \varepsilon \mathfrak{P}_2 \perp \mathfrak{H}$, then

$$(5.11) \qquad \omega = \gamma + \epsilon, \quad \delta\gamma = d\epsilon = 0, \quad \gamma \varepsilon \mathfrak{P}_2^+, \quad \epsilon \varepsilon \mathfrak{P}_2^-, \quad (\gamma, \epsilon) = 0.$$

Hence, from (5.9) and Theorem 5.2, we see that

$$D(\omega) = D(\gamma) + D(\epsilon) \geqq \lambda^+ \| \gamma \|^2 + \lambda^- \| \epsilon \|^2 \geqq \lambda[2 \| \gamma \|^2 + 2 \| \epsilon \|^2]$$
$$\geqq \lambda \| \omega \|^2, \lambda = \min[\lambda^+/2, \lambda^-/2].$$

The following theorem completes the analogy with the case for a compact manifold without boundary.

THEOREM 5.10. *If* $\eta \varepsilon \mathfrak{L}_2$ *and is* $\mathfrak{L}_2$*-orthogonal to* $\mathfrak{H}$, *there is a unique form* Ω *in* $\mathfrak{P}_2$ *and* $\mathfrak{L}_2$*-orthogonal to* $\mathfrak{H}$ *such that*

$$(5.12) \qquad (d\Omega, d\zeta) + (\delta\Omega, \delta\zeta) = (\eta, \zeta), \quad \zeta \varepsilon \mathfrak{P}_2.$$

Moreover,

$$(5.13) \qquad d\Omega = d\Omega^+ \text{ and } \delta\Omega = \delta\Omega^-,$$

Ω^+ *and* Ω^- *being the respective plus and minus potentials of* η. *The differentiability properties of* Ω *are the same as those in Theorem* 5.4.

Proof. The proof that Ω exists in $\mathfrak{P}_2$ and is unique is just like that of Theorem 5.3. Obviously $\eta \varepsilon \mathfrak{L}_2 \ominus \mathfrak{H}^+$ and $\mathfrak{L}_2 \ominus \mathfrak{H}^-$ so that its plus and minus potentials exist. Accordingly we have, for example,

$$(5.14) \qquad (d\Omega - d\Omega^+, d\zeta) + (\delta\Omega - \delta\Omega^+, \delta\zeta) = 0 \text{ for all } \zeta \varepsilon \mathfrak{P}_2^+.$$

But, from Theorem 5.6, we may find a $\zeta \varepsilon \mathfrak{P}_2^+$ such that

$$(5.15) \qquad d\zeta = d\Omega - d\Omega^+, \delta\zeta = 0.$$

Using (5.14 and (5.15) and a similar argument for Ω^-, we derive (5.13).

Now, let us consider the decomposition (5.7) for Ω, Ω^+, and Ω^-. Using (5.13) and the last statement in Theorem 5.8 also, we obtain

$$(5.16) \qquad \begin{gathered} \Omega = \Gamma + E, \quad \Omega^\pm = H^\pm + \Gamma^\pm + E^\pm, \quad \Omega = \Omega^\pm + K^\pm, \quad dK^+ = \delta K^- = 0 \\ K^+ = H_1^+ + E_1^+, \quad K^- = H_1^- + \Gamma_1^-. \end{gathered}$$

where the Γ's $\varepsilon\, \mathfrak{C}$, the E's $\varepsilon\, \mathfrak{D}$, and the H's $\varepsilon\, \mathfrak{H}$. From (5.16) and the uniqueness of the decomposition, we obtain

$$
(5.17)\qquad \begin{gathered} H^{\pm}+H_1^{\pm}=0, \quad \Gamma+E=\Gamma^{+}+(E^{+}+E_1^{+})=(\Gamma^{-}+\Gamma_1^{-})+E^{-} \\ \Gamma=\Gamma^{+}, \quad E=E^{-}. \end{gathered}
$$

The differentiability properties of $d\Omega$ and $\delta\Omega$ follow from (5.13) and Theorem 5.4 and those for Ω follow from (5.16) and (5.17) and Theorems 5.4 and 5.9.

Remark. We cannot conclude the differentiability of Ω directly from (5.12) and Theorem 3.4, since the equations (5.12) are not the same as (3.1) and (3.3) since the boundary integral corresponding to the first term in (4.8) is not necessarily zero in this case.

Definition 5.4. The form Ω in Theorem 5.10 is called the *potential of* η.

Important Remark. All differentiability properties at either interior or boundary points are local; all differentiability results extend immediately to cases where the given hypotheses hold in some coordinate patch, the conclusions then holding in that patch.

6. Boundary value problems. In this section, we derive briefly the results concerning boundary value problems for harmonic forms and fields which have been obtained by other methods by Duff and Spencer [3] and Conner [1]. The differentiability results on the interior have been obtained in part I and stated also in [8]; the corresponding results on the boundary depend on the given boundary values as well as on the differentiability of $\mathfrak{M}$ and are stated below.

The following theorem is seen (from their proofs) to be equivalent to Theorems 3 and 4 of [3] (pp. 150, 151):

THEOREM 6.1. (a) *If ω is any closed form in $\mathfrak{P}_2$, there is a unique harmonic field H such that* ($r =$ *degree of* ω)

$$
\omega = H + d\beta, \; tH = t\omega, \beta, d\beta \,\varepsilon\, \mathfrak{P}_2^{-} (t\beta = td\beta = 0), \quad 0 \leqq r \leqq n-1.
$$

(b) *If ω is any co-closed form in $\mathfrak{P}_2$, there is a unique harmonic field H such that*

$$
\omega = H + \delta\alpha, \; nH = n\omega, \alpha, \delta\alpha \,\varepsilon\, \mathfrak{P}_2^{+} (n\alpha = n\delta\alpha = 0), \quad 1 \leqq r \leqq n.
$$

In either case, the differentiability results are as follows:

(i) *If $\mathfrak{M}$ is of class C_1^1 and $\omega \,\varepsilon\, \mathfrak{P}_{2\lambda}$, $\lambda = \rho - 1 + \mu$, $0 < \mu < 1$, then H is also and ω and $H \,\varepsilon\, C_\mu^0$ at B. If $r = 0$, ω and H are constants.*

(ii) *If* $\mathfrak{M}$ *is of class* $C_\mu{}^k$ (C^∞, *analytic*) *and* ω *is of class* $C_\mu{}^{k-1}$ (C^∞, C^ω), $k \geqq 2$, $0 < \mu < 1$, *then* H *is of class* $C_\mu{}^{k-1}$ (C^∞, *analytic*).

Proof. If $d\omega = 0$, then from Theorems 5.7 and 5.8, we see that the term $\delta\alpha = 0$ in the decomposition which proves (a); (b) is proved similarly. The differentiability results follow from Theorem 5.8.

The next theorem is a refinement of Theorem 2 of [3]:

THEOREM 6.2. *If* η *is any form in* $\mathfrak{P}_2$, *there is a form* ω *in* $\mathfrak{P}_2$ *such that* $t\omega = t\eta$ *and* $d\omega$ *is a harmonic field. If, also,* $\eta = \delta\chi$ *for some* χ *in* $\mathfrak{P}_2$, *then there is a unique* ω *of the form* $\omega = \delta\xi$ *with* ξ *in* $\mathfrak{P}_2$ *which satisfies the conditions above.*

Proof. Let H^- be the projection of $d\eta$ on $\mathfrak{H}^-$, let α be the minus potential of $d\eta - H^-$, and let

$$\omega = \eta - \gamma, \quad \gamma = \delta\alpha, \quad \epsilon = d\alpha. \tag{6.1}$$

Then γ and ϵ are in $\mathfrak{P}_2{}^-$ and from (5.3) ($\alpha = \Omega$), we have

$$t\gamma = 0 = t\epsilon, \quad d\gamma + \delta\epsilon = d\eta - H^-, \quad d\omega = d\eta - d\gamma = H^- + \delta\epsilon. \tag{6.2}$$

From (6.2) and the last statement in Theorem 5.8, we see that

$$d\omega \,\varepsilon\, (\mathfrak{H} \oplus \mathfrak{D}) \cap (\mathfrak{H} \oplus \mathfrak{C}) = \mathfrak{H}.$$

Now, suppose $\eta = \delta\chi$ for some χ in $\mathfrak{P}_2$. Then $\omega = \delta(\chi - \alpha)$ from (6.1). Suppose ω_1 also satisfies all these conditions. Then

$$t(\omega - \omega_1) = 0, \quad \omega - \omega_1 = \delta\nu \ (\nu = \chi - \alpha - \xi_1), \quad d(\omega - \omega_1) \,\varepsilon\, \mathfrak{H}.$$

But then, from the definitions of $\mathfrak{C}$ and $\mathfrak{D}$ and Theorem 5.8, we obtain

$$\omega - \omega_1 \,\varepsilon\, \mathfrak{P}_2{}^-, \ \therefore d(\omega - \omega_1) \,\varepsilon\, \mathfrak{D} \cap \mathfrak{H}, \ \therefore d(\omega - \omega_1) = 0, \ \therefore \omega - \omega_1 \,\varepsilon\, \mathfrak{H} \oplus \mathfrak{D},$$

$$\omega - \omega_1 = \delta\nu \,\varepsilon\, \mathfrak{H} \oplus \mathfrak{C}, \ \therefore \omega - \omega_1 \,\varepsilon\, \mathfrak{H}^-, \ \omega - \omega_1 \,\varepsilon\, \mathfrak{L}_2 - \mathfrak{H}^- \quad \text{(Theorem 5.5(i))},$$

$$\therefore \omega - \omega_1 = 0.$$

We now consider boundary value problems for harmonic forms as distinct from harmonic fields. We begin by defining

$$\mathfrak{P}_{20} = \mathfrak{P}_2{}^+ \cap \mathfrak{P}_2{}^-, \quad \mathfrak{H}_0 = \mathfrak{H}^+ \cap \mathfrak{H}^- = \mathfrak{H} \cap \mathfrak{P}_{20}.$$

Then $\mathfrak{P}_{20}$ consists of all $\mathfrak{P}_2$ forms ω with $t\omega = n\omega = 0$. The last part of the following theorem is due to Spencer [11].

THEOREM 6.3. *$\mathfrak{H}_0$ is finite dimensional. If $H \varepsilon \mathfrak{H}_0$, $\mathfrak{H}^+$, or $\mathfrak{H}^-$, then H has the differentiability properties on $\mathfrak{B}$ stated in Theorem 5 of [8]. If $\mathfrak{M}$ (and $\mathfrak{B}$) is analytic and $H \varepsilon \mathfrak{H}_0$, then $H = 0$.*

Proof. The first statement is a consequence of Theorem 5.1. Obviously

$$(dH, d\zeta) + (\delta H, \delta\zeta) = 0 \tag{6.3}$$

for all ζ in $\mathfrak{P}_2$ and hence in $\mathfrak{P}_2^+$ or $\mathfrak{P}_2^-$. If $H \varepsilon \mathfrak{P}_2^+(\mathfrak{P}_2^-)$, then (6.3) is equivalent to (3.1) and (3.3) so the differentiability results follow.

Now chose any point P on $\mathfrak{B}$ and choose an analytic admissible boundary coordinate system mapping the origin into P and $G_R \cup \sigma_R$ into a neighborhood of P with $g_{ij}(0) = \delta_{ij}$. Since $H \varepsilon \mathfrak{P}_2^+$, (6.3) is equivalent to (3.1) and (3.3) with a, b, b^*, and c analytic and $e = f = 0$. Accordingly H is analytic on $x^n = 0$ and hence H and the coefficients can all be extended analytically across $x^n = 0$. Since $dH = \delta H = 0$ we see that all the $H_{(i)}$ and $H_{(i)x^n} = 0$ on $x^n = 0$. The result follows from the Cauchy-Kowalewsky theorem.

Definition 6.1. A form K is *harmonic* on $\mathfrak{M}$ if and only if K, dK, and $\delta K \varepsilon \mathfrak{P}_2$ on any damain interior to $\mathfrak{M}$ with $\delta dK + d\delta K = 0$ there.

THEOREM 6.4. *If $\omega \varepsilon \mathfrak{P}_2$, there exists a harmonic form K in $\mathfrak{P}_2$ such that $tK = t\omega$, $nK = n\omega$. Any two such solutions differ by a harmonic field in $\mathfrak{H}_0$. The differentiability results for K are the same as those in Theorem 6.1 except that in the case of zero-forms, $K \varepsilon C_\mu^k$ if $\omega \varepsilon C_\mu^k$.*

Proof. Write $K = \omega + \eta$ and minimize

$$D(\omega + \eta) = D(\eta) + 2(d\eta, d\omega) + 2(\delta\eta, \delta\omega) + D(\omega)$$

among all η in $\mathfrak{P}_{20} \cap (\mathfrak{L}_2 \ominus \mathfrak{H}_0)$. Then $D(\eta) \geqq \lambda \| \eta \|^2$ so that the minimizing function exists as usual. Then K is easily seen to satisfy (6.3) for all ζ in $\mathfrak{P}_{20}$ so that K is harmonic on the interior of $\mathfrak{M}$ (using the differentiability results of Section 4 and 7 of part I). Since $\eta \varepsilon \mathfrak{P}_{20} \cap (\mathfrak{L}_2 \ominus \mathfrak{H}_0)$, the equivalent equation

$$(d\eta + d\omega, d\zeta) + (\delta\eta + \delta\omega, \delta\zeta) = 0 \tag{6.4}$$

for all $\zeta \varepsilon \mathfrak{P}_{20}$ is of the form (3.1) and (3.3) on boundary coordinate patches. The differentiability results follow from Section 3. In case $K \varepsilon \mathfrak{P}_{20}$, we may set $H = \zeta = K$ in (6.3) and conclude that K is a harmonic field in $\mathfrak{H}_0$.

LEMMA 6.1. *Suppose $f(x)$ is of class $\mathfrak{P}_2$ for $|x| < R$ with support interior to that sphere. Then there is a function $u(x^1, \cdots, x^n, h)$ with*

support in $|x|^2 + h^2 < R^2$, *with* u *and* $\nabla u \varepsilon \mathfrak{P}_2$ *there, with* $u(x,0) = 0$, $u_h(x,0) = f(x)$. *If, also* $f(x) \varepsilon C_\mu^k$, $0 \leqq \mu \leqq 1$, *then* u *may be taken to be of class* C_μ^{k+1} *there. If, also,* $f \varepsilon C^\infty$, *then* u *may be taken to be of class* C^∞. *If, also,* f *is analytic near* x_0, *then* u *may be taken to be analytic in* (x,h) *near* $(x_0, 0)$.

Proof. (By Friedrichs mollifier): Let $k(x)$ be of class C^∞ for all x and have support in $|x| < 1$ with the integral of $k(x) = 1$. Extend $f(x)$ to be zero outside $|x| = R$ and define

$$u_1(x,h) = h \int_{|y|<1} f(x+hy)k(y)\,dy = h^{1-n} \int_{-\infty}^{\infty} f(z)k(z/h - x/h)\,dz. \tag{6.4}$$

In all cases u_1 is of class C^∞ in (x,h) for $h \neq 0$ and we have

$$\begin{aligned} u_{1x^a}(x,h) &= -\int_{B(0,1)} f(x+h\eta)k_{x^a}(\eta)\,d\eta \\ u_{1h}(x,h) &= -\int_{B(0,1)} f(x+h\eta)[\eta^a k_{x^a}(\eta) + (n-1)k(\eta)]\,d\eta \end{aligned} \tag{6.5}$$

and $u_{1h}(x,h)$ tends to $f(x)$ as $h \to 0$ for each x for which the Lebesgue derivative of the integral of $f = f(x)$. It is easy to see from (6.5) that u_1 has the desired differentiability properties. Since $f = 0$ for all x on and near $\partial\sigma_R$, we may multiply u_1 by a function $l(h)$ of h alone which is analytic at $h = 0$ with $l(0) = 1$ and $l'(0) = 0$ and C^∞ for all h and zero for $|h| \geqq h_0$ where h_0 is chosen small enough so that $u(x,h) = l(h)u_1(x,h)$ has support in $|x|^2 + h^2 < R^2$.

The following lemma simplifies the statements and proofs of our remaining results on boundary value problems.

LEMMA 6.2. (i) *Suppose* ξ *and* η *are* r *and* $r+1$ *forms on* $\mathfrak{M}$, *respectively, with* ξ, $d\xi$, *and* η *in* $\mathfrak{P}_2$ *along* $\mathfrak{B}$ (*i.e.* $b\xi$, $bd\xi$, $b\eta \varepsilon \mathfrak{P}_2$ *along* $\mathfrak{B}$). *Then there exists an* r *form* ω *with* ω *and* $d\omega$ *in* $\mathfrak{P}_2$ *on* $\mathfrak{M}$ *with*

$$n\omega = n\xi, \quad nd\omega = n\eta \text{ on } \mathfrak{B}. \tag{6.6}$$

If ξ, $d\xi$, *and* $\eta \varepsilon \mathfrak{P}_{2\lambda}$ *along* $\mathfrak{B}$, ω *may be chosen so that* ω *and* $d\omega \varepsilon \mathfrak{P}_{2\lambda}$ *on* $\mathfrak{M}$. *If* $\mathfrak{M}$ *is of class* C_μ^k *and* ξ, $d\xi$, *and* $\eta \varepsilon C_\mu^{k-1}$ *along* $\mathfrak{B}$ ($k \geqq 2$, $0 < \mu < 1$), ω *may be chosen so that* ω *and* $d\omega \varepsilon C_\mu^{k-1}$. *If* $\mathfrak{M} \varepsilon C^\infty$ *and* ξ *and* $\eta \varepsilon C^\infty$ *along* $\mathfrak{B}$, ω *may be chosen to be of class* C^∞ *on* $\mathfrak{M}$; *if* $\mathfrak{M}$ *is analytic and* ξ *and* η *are also analytic near a point* P *of* $\mathfrak{B}$, *then* ω *may be chosen to be also analytic on* $\mathfrak{M}$ *near* P. *If* $r = 0$, *the first condition is vacuous and we may choose* $\omega \varepsilon C_\mu^k$

if $\mathfrak{M} \varepsilon C_\mu^k$ *and* $\eta \varepsilon C_\mu^{k-1}$ *or to be in* $\mathfrak{P}_2$ *with all the derivatives of its components being in* $\mathfrak{P}_2$ *if* $\mathfrak{M} \varepsilon C_1^1$. *If* $r = n$, *the second condition is vacuous and* $n\omega = b\omega$, $n\xi = b\xi$, *and the result is that of Lemma* 4.1.

(ii) *The dual of* (i), *with*

$$t\omega = t\xi,\ t\delta\omega = t\eta \text{ on } \mathfrak{B}. \tag{6.7}$$

(iii) *Suppose* ξ *and* η *are* $r+1$ *and* $r-1$ *forms on* $\mathfrak{M}$, *respectively, with* ξ *and* η *in* $\mathfrak{P}_2$ *along* $\mathfrak{B}$. *Then there exist a form* ω *with* ω, $d\omega$, *and* $\delta\omega$ *in* $\mathfrak{P}_2$ *on* $\mathfrak{M}$ *such that*

$$nd\omega = n\xi,\ t\delta\omega = t\eta \text{ on } \mathfrak{B}; \tag{6.8}$$

if ξ *and* $\eta \varepsilon \mathfrak{P}_{2\lambda}$ *along* $\mathfrak{B}$, ω *may be chosen so that* ω, $d\omega$, *and* $\delta\omega \varepsilon \mathfrak{P}_{2\lambda}$ *on* $\mathfrak{M}$. *If* $\mathfrak{M}$ *is of class* C_μ^k $(k \geqq 2, 0 < \mu < 1)$ *and* ξ *and* $\eta \varepsilon C_\mu^{k-1}$ *along* $\mathfrak{B}$, *then* ω *may be chosen so that* ω, $d\omega$, *and* $\delta\omega \varepsilon C_\mu^{k-1}$ *on* $\mathfrak{M}$. *If* $\mathfrak{M} \varepsilon C^\infty$ *and* ξ *and* $\eta \varepsilon C^\infty$ *along* $\mathfrak{B}$, *then* ω *may be taken in* C^∞; *if also* $\mathfrak{M}$, ξ, *and* η *are analytic near a point* P *on* $\mathfrak{B}$, *then* ω *may be taken analytic on* $\mathfrak{M}$ *near* P. *If* $r = 0$, *the condition on* $\delta\omega$ *is vacuous and* ω *may be taken to have one additional degree of differentiability (but not* $d\omega$*); the case* $r = n$ *is the dual of this* ($^*\omega$ *additionally differentiable*).

Proof of (i). From the proof of Lemma 4.1, we conclude that there are forms $\xi_1, \cdots, \xi_S$ and $\eta_1, \cdots, \eta_S$ on $\mathfrak{M}$ in which each pair (ξ_s, η_s) has support in some one admissible coordinate patch and each satisfies the differentiability requirements of ξ and η but on $\mathfrak{M}$. Thus our problem is reduced to the case where ξ and η have their stated properties on $\mathfrak{M}$ and support in an admissible boundary coordinate system with domain G_R. We write $\omega = \xi + \omega_1$, $\eta = d\xi + \eta_1$ and seek an ω_1 with the desired differentiability properties satisfying

$$n\omega_1 = 0,\ nd\omega_1 = \eta_1 \text{ on } \mathfrak{B}. \tag{6.9}$$

Actually, we may set $b\omega_1 = 0$ in which case (6.9) is seen using the formula for $d\phi$ (see formula 1.9 of [3]) to reduce to

$$(-1)^r \omega_{1i_1\cdots i_r x^n}(0, x'_n) = \eta_{1i_1\cdots i_r},\ i_r < n,\ \omega_{1i_1\cdots i_{r-1}n}(0, x'_n) = 0. \tag{6.10}$$

From Lemma 6.1, we see that we extend the components to be of class C_μ^k (or etc.) in G_R and to vanish on and near S_R^-. Accordingly ω_1 and $d\omega_1$ have the proper class on $\mathfrak{M}$.

Proof of (ii). We begin by taking the duals of all the forms and then proceeding as in (i).

Proof of (iii). From (i) and (ii) we may find forms ω_1 and ω_2 with ω_1, $d\omega_1$, ω_2, and $\delta\omega_2$ all in C_μ^{k-1} (or etc.) such that

$$n\omega_1 = 0,\ nd\omega_1 = n\xi,\ t\omega_2 = 0,\ t\delta\omega_2 = t\eta \text{ on } \mathfrak{B}.$$

Then from Theorem 5.6(ii), we may find forms ω_3 and ω_4 with

$$n\omega_3 = 0,\ d\omega_3 = d\omega_1,\ \delta\omega_3 = 0,\ t\omega_4 = 0,\ \delta\omega_4 = \delta\omega_2,\ d\omega_4 = 0.$$

The desired $\omega = \omega_3 + \omega_4$. It is seen from Theorems 5.4, 5.5, 5.6, and their proofs and that of hTeorem 5.8, etc., that ω has the desired differentiability properties.

LEMMA 6.3. *Suppose* $\omega \varepsilon \mathfrak{L}_2 \ominus \mathfrak{H}$, $\omega \varepsilon \mathfrak{P}_2$ *on all domains* (*with closures*) *interior to* $\mathfrak{M}$, *and* $D(\omega)$ *is finite. Then* $\omega \varepsilon \mathfrak{P}_2$ *on* $\mathfrak{M}$.

Proof. We may choose a finite covering of $\mathfrak{M}$ by admissible coordinate patches, choose a corresponding partition of unity, and thus express $\omega = \omega_1 + \cdots + \omega_S$, each ω_s having support in some one coordinate patch, being in $\mathfrak{P}_2$ on interior domains, in $\mathfrak{L}_2$ on $\mathfrak{M}$, with $D(\omega_s)$ finite; each ω_s with support interior to $\mathfrak{M} \varepsilon \mathfrak{P}_2$. Now, consider an ω_s with support in a boundary patch with domain G_R; we assume $\omega_s = 0$ elsewhere. Now for each p define $\omega_s{}^p$ on G_R by

$$\omega_{s(i)}{}^p(x^n, x'_n) = \omega_{s(i)}(x^n - h_p, x'_n) \qquad h_p = R\cdot(p+1)^{-1}.$$

If we define $'\omega_s{}^p = \omega_s$ for those with interior support and define $'\omega^p = \omega_1{}^p + \cdots + \omega_S{}^p$ on $\mathfrak{M}$ we see that $'\omega^p$, $d'\omega^p$, and $\delta'\omega^p$ tend strongly in $\mathfrak{L}_2$ on $\mathfrak{M}$ to ω, $d\omega$, and $\delta\omega$, each $'\omega^p$ being in $\mathfrak{P}_2$. If we let ω^p be the projection of $'\omega^p$ on $\mathfrak{L}_2 \ominus \mathfrak{H}$, we see from Theorem 5.8 that we may approximate equally well using $\{\omega^p\}$. But then, from Theorem 5.9 and the fact that $D(\omega^p - \omega) \to 0$, we see that $\| \omega^p \|$ is uniformly bounded. Hence a subsequence, still called ω^p, converges weakly in $\mathfrak{P}_2$ and hence strongly in $\mathfrak{L}_2$ to something which must be ω.

We can now state the results of Conner [1]:

THEOREM 6.5. *In each part of Lemma* 6.2, *the form* ω *may be replaced by a harmonic form* K *with the same differentiability properties. In fact if in* (i) *we merely require that* ξ, $d\xi$, *and* $\eta \varepsilon \mathfrak{P}_2(\mathfrak{P}_{2\lambda})$ *on* $\mathfrak{M}$, *then any such* $K \varepsilon \mathfrak{P}_2(\mathfrak{P}_{2\lambda})$ *along with* dK; *analogous results hold in cases* (ii) *and* (iii), *except that in* (iii) K *must be properly chosen, orthogonal to* $\mathfrak{H}$ *for example. In case* (i)((ii)), *any two solutions* K *with* K *and* $dK(\delta K)$ *in* $\mathfrak{P}_2$ *differ by a harmonic field in* $\mathfrak{H}^+(\mathfrak{H}^-)$; *in case* (iii), *any two solutions* K *in* $\mathfrak{L}_2$ *with*

dK and δK in $\mathfrak{L}_2$ differ by a harmonic field. If in (i) ((ii)) $\delta\omega(d\omega) \varepsilon C_\mu^{k-1}$ *(or etc.), then $\delta K(dK)$ does also.*

Proof of case (i). Let ω_1 be the positive potential of $\delta d\omega$; since $d\omega \varepsilon \mathfrak{P}_2^+$, $\delta d\omega \varepsilon \mathfrak{L}_2 \ominus \mathfrak{H}$. Choose ω_2 in $\mathfrak{P}_2^+$ (Theorem 5.6) so that

$$d\omega_2 = 0, \quad \delta\omega_2 = \delta\omega. \tag{6.11}$$

Let us define K by

$$K = \omega - \omega_1 - \omega_2. \tag{6.12}$$

Then we have

$$dK = d\omega - d\omega_1 \varepsilon \mathfrak{P}_2, \ \delta dK = \delta d\omega - \delta d\omega_1 = d\delta\omega_1, \ nK = n\omega, \ ndK = nd\omega. \tag{6.13}$$

Then if $\zeta \varepsilon \mathfrak{P}_{20}$, we have

$$(dK, d\zeta) + (\delta K, \delta\zeta) = (\delta dK, \zeta) + (\delta K, \delta\zeta)$$
$$= (d\delta\omega_1, \zeta) + (\delta K, \delta\zeta) = (\delta K + \delta\omega_1, \delta\zeta) = (\delta\omega - \delta\omega_2, \delta\zeta) = 0$$

using (6.11). Hence K is harmonic from Sections 4 and 7, part I. If $\delta\omega$ were already known to be in $C_\mu^{k-1}(\mathfrak{P}_2$, etc.), we would simply have defined K by

$$K = \omega - \Omega \tag{6.14}$$

where Ω is the plus potential of $\Delta\omega$.

Now, in the case of two solutions, the *difference* K' would be a harmonic form with K' and dK' in $\mathfrak{P}_2$ and

$$nK' = ndK' = 0. \tag{6.15}$$

Now $\delta K' \varepsilon \mathfrak{P}_2$ on interior domains is harmonic there (Sections 4 and 7, part I) and is in $\mathfrak{L}_2$ on $\mathfrak{M}$. Moreover

$$d\delta K' = -\delta dK' \varepsilon \mathfrak{L}_2, \quad \delta\delta K' = 0.$$

Hence $D(\delta K')$ is finite and, by Definition 5.3, $\delta K' \varepsilon \mathfrak{C} \subset \mathfrak{L}_2 \ominus \mathfrak{H}$. Hence $\delta K'$ is also in $\mathfrak{P}_2$ by Lemma 6.3. But then

$$(dK', dK') + (\delta K', \delta K') = (K', \Delta K') + \int_{\mathfrak{B}} K' \cap {}^*dK' - \delta K' \cap {}^*K' = 0, \tag{6.16}$$

so that $K' \varepsilon \mathfrak{H}^+$.

Proof of case (ii). Dual to case (i).

Proof of case (iii). In this case we define K by (6.14) and note that it satisfies the conditions if we choose Ω as the potential of

$$\Delta\omega = \delta(d\omega) + d(\delta\omega)\, \varepsilon\, \mathfrak{C} \oplus \mathfrak{D}$$

(Definition 5.3, Theorem 5.10).

Suppose now that we have two solutions with the stated properties. Then the projection K' of the difference on $\mathfrak{C} \oplus \mathfrak{D}$ is harmonic and in $\mathfrak{L}_2 \ominus \mathfrak{H}$ with dK' and $\delta K'$ in $\mathfrak{P}_2$ with $ndK' = t\delta K' = 0$. Then (6.16) holds so that $K' \,\varepsilon\, \mathfrak{H}$. Hence $K' = 0$.

Remarks. We conclude with remarks about a result of Spencer on what he called a bounded manifold [10]. An open manifold $\mathfrak{M}$ is *bounded* if and only if there is a compact manifold $\mathfrak{M}_1$, with or without boundary, of which $\mathfrak{M}$ is an open subdomain with closure interior to $\mathfrak{M}_1$; as usual, we assume $\mathfrak{M}_1$ of class $C_1{}^1$ at least. We may define the subspace $\mathfrak{P}_{20}$ of $\mathfrak{P}_2$ forms on $\mathfrak{M}$ as the closure in $\mathfrak{P}_2$ on $\mathfrak{M}_1$ of the $\mathfrak{P}_2$ (or Lipschitz) forms with support interior to $\mathfrak{M}$; any form in $\mathfrak{P}_{20}$ is in $\mathfrak{P}_2$ on $\mathfrak{M}_1$ if it is defined to be zero on and outside $\partial\mathfrak{M}$. Thus, if $\mathfrak{N}$ is any finite covering of $\mathfrak{M}_1$, we have

$$D(\omega) \geqq k \parallel \omega \parallel^2 - l(\omega, \omega)$$

for all ω in $\mathfrak{P}_{20}$ on $\mathfrak{M}$ the norm corresponding to $\mathfrak{N}$. Clearly $D(\omega)$ is still lower-semicontinuous with respect to weak convergence in $\mathfrak{P}_{20}$ and weak convergence in $\mathfrak{P}_{20}$ implies strong convergence in $\mathfrak{L}_2$. Hence we may generalize Lemma 5.1 and Theorems 5.1 to 5.3 immediately and hence also Theorems 6.3 and 6.4 except for the differentiability on the boundary; the conditions $tK = t\omega$ and $nK = n\omega$ of Theorem 6.4 must be changed to $K - \omega \,\varepsilon\, \mathfrak{P}_{20}$. In Theorem 6.3, if $\mathfrak{M}_1$ is analytic, we see that any harmonic field in $\mathfrak{P}_{20}$ is also one on $\mathfrak{M}_1$ if extended as above; since $\mathfrak{M} \cup \partial\mathfrak{M}$ is interior to $\mathfrak{M}_1$, it follows that any such field is zero. Moreover if K is a harmonic *form* [defined as in Definition 6.1] in $\mathfrak{P}_{20}$, we see (by first considering ζ's with support interior to $\mathfrak{M}$) that $(dK, d\zeta) + (\delta K, \delta\zeta) = 0$ for all ζ in $\mathfrak{P}_{20}$; by taking $\zeta = K$, we see that K is a harmonic field.

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FIBRE SYSTEMS OF JACOBIAN VARIETIES.*[1]

By Jun-ichi Igusa.

The method of Picard and Poincaré in studying an algebraic surface V consists in applying the theory of curves to the curves of an auxililary linear pencil $\{C_u\}$ on V. The same method was employed later by Lefschetz in his theory of algebraic surfaces (cf. [19], Chapters 6-7). If we examine closely their method, we can see that they considered implicitly a certain variety $\mathcal{J}$ which the author proposes to call the *Néron variety of V associated with* $\{C_u\}$. In fact Néron was the first who considered $\mathcal{J}$ explicitly in his algebraic proof of the theorem of the base [12]. The variety $\mathcal{J}$ is the graph of the correspondence $u \rightarrow J_u$ between u and the Jacobian variety J_u of C_u. Here, we restrict our attention to such linear pencils whose members are all irreducible and whose general members are nonsingular. If V does not carry any multiple curve, we can always find such a linear pencil. Also, if V is nonsingular, we can assume that singular members of the pencil are curves with ordinary double points. Now, the main part of the paper is devoted to determining the "degenerate fibres" of the fibre system $\{u \times J_u\}$ on $\mathcal{J}$ at those finite number of values of u where C_u become singular. We note that in Néron's case such degenerate fibres are not considered explicitly. The same thing can be said about Chow's investigations on Abelian varieties over function fields [4]. However, in some problems in algebraic geometry it becomes necessary to consider those degenerate fibres and also the behavior of fibres along the degenerate fibres. We shall show that the degenerate fibres are certain completions of the generalized Jacobian varieties of the singular curves in the sense of Rosenlicht [15]. The singular locus of $\mathcal{J}$ is contained in the union of singular loci of degenerate fibres. Also we can define in a natural way a birational map ϕ from V into $\mathcal{J}$, and we can show that ϕ gives isomorphisms of the Albanese varieties and of the spaces of linear differential forms of the first kind of V and $\mathcal{J}$. This result has already been applied to show that the dimension of the Albanese variety of an arbitrary

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variety is not greater than the dimension of the space of linear differential forms of the first kind on the variety [6].

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1. Existence of general linear pencils.

1. Let V be an arbitrary algebraic variety in a projective space P^n. We shall denote by $\mathfrak{K}$ the universal domain of our algebraic geometry.[2] Let H be a hypersurface of order m in P^n not containing V. Then the intersection product $V \cdot H$ is defined, and $V \cdot H$ is a P^n-cycle carried by V. We shall denote by $\mathfrak{L}_m$ the totality of such cycles. Now the system of hypersurfaces of order m in P^n defines a biregular rational map ϕ of P^n into a projective space of dimension $C_n{}^{m+n} - 1$. Consequently V is transformed biregularly to an algebraic variety V^* in this projective space. We shall denote by P^N the smallest subspace containing V^*. If L is a hyperplane in P^N and if T is the graph of $\mathrm{Tr}_V \phi$, then $\mathrm{pr}_V[T \cdot (V \times L)]$ is a member of $\mathfrak{L}_m$. Moreover this operation defines a one-to-one correspondence between the system of hyperplanes in P^N and $\mathfrak{L}_m$. The system of hyperplanes in P^N can be identified with the dual space P'^N of P^N. If P'^r is a subspace of P'^N and if X_0 is a P^n-cycle of dimension $\dim(V) - 1$ carried by V, then the set of P^n-cycles of the form $X + X_0$, where X vary in P'^r, is called a *linear system of dimension r on V*. In particular $\mathfrak{L}_m$ itself is a linear system of dimension N on V. In the following V is either an algebraic surface or an algebraic curve.

Now let V be an arbitrary algebraic surface. We know in general that of all fields in $\mathfrak{K}$ which define V, there is one smallest one k_0. We shall denote its algebraic closure by k. Since the rational map ϕ is defined over the prime field, k_0 is again the smallest field of definition of V^*. Therefore the projective space P^N is defined over k_0. A positive cycle is called a reducible variety if the sum of the coefficients of its components is at least equal to 2. A normally algebraic bunch over a field will be called, after Zariski, a closed set over this field. We shall now prove the following theorem.[3]

[2] We shall use the results and terminology of Weil's book [16] mostly without quoting.

[3] A slightly less general result was proved by Néron and Samuel [13]. We shall use their idea in our proof.

THEOREM 1. *The set* $\mathfrak{C}_1$ *of hyperplanes in* P^N, *which have reducible curves as intersection products with* V^*, *is a closed set over* k_0 *of dimension at most* $N-2$ *for* $m \geqq 3$. *The only exceptional surface for* $m=2$ *is a subspace* P^2 *of* P^n.

Proof. The assertion that $\mathfrak{C}_1$ is a closed set over k_0 can be proved by the standard method [1], hence we can omit it here. We shall first treat the case where V is a subspace of P^n. Let $G(X)$ and $H(X)$ be homogeneous polynomials in three letters X_0, X_1, X_2 of orders α and β respectively such that $\alpha+\beta=m$. Put $F(X)=G(X)H(X)$. The set of all $F(X)$ of this form is an algebraic variety of co-dimension $C_2{}^{m+2}-(C_2{}^{\alpha+2}+C_2{}^{\beta+2})+1 = \alpha(m-\alpha)$ in the projective space of homogeneous polynomials of order m. If we remark that the minimal value of $\alpha(m-\alpha)$ for $1 \leqq \alpha \leqq m-1$ is $m-1$, the first assertion and the trivial part of the second assertion for $V=P^2$ follows immediately. We shall exclude this case in the following. We fix one m by $m \geqq 2$. Let R be a component of $\mathfrak{C}_1$. Then R is defined over k, and we have $\dim(R) \leqq N-1$ by a theorem of Bertini [21, 9]. We shall derive a contradiction from the assumption $\dim(R)=N-1$. Let u be a generic point of R over k and let L_u be the corresponding hyperplane in P^N. Then $V^* \cdot L_u$ is a rational P^N-cycle over $k(u)$, hence it determines a rational positive cycle X of the product $P^N \times R$ over k, every component of which has the projection R on R and such that $X(u)=V^* \cdot L_u$. Let $x^* \times u$ be a generic point of some component of X over k. Then x^* is a generic point of V^* over k. Otherwise x^* has a locus C^* over k, which is contained in every hyperplane of R. However, this is impossible by the following reason: Let C be the corresponding curve on V. Then we have $\mathrm{ord}(C^*)=m \cdot \mathrm{ord}(C) \geqq 2$. Therefore C^* contains three points which are not colinear. Hence the hypersurface R is contained in the intersection of dual hyperplanes of these three points; a contradiction. Let C^* be a component of $V^* \cdot L_u$ containing x^*. Then C^* appears with coefficient one in $V^* \cdot L_u$, and it is the only component of $V^* \cdot L_u$ passing through x^*. Otherwise, by a criterion of multiplicity one [16, p. 141] L_u contains the tangent plane of V^* at x^*. The totality of hyperplanes in P^N having this property is a subspace T of P'^N of dimension $N-3$ defined over $k_0(x^*)$. On the other hand if K is the algebraic closure of $k_0(x^*)$, the point u has a locus S of dimension $N-2$ over K, and every hyperplane of S contains the tangent plane of V^* at x^*, i.e., S is contained in T; a contradiction. Since C^* appears with coefficient one in $X(u)$, it is defined over a separably algebraic extension of $k(u)$. Since C^* is also the only component of $X(u)$ passing through x^*, we conclude that C^* is defined

over $k(u,x^*)$, hence over $K(u)$. Therefore, by assumption $V^*\cdot L_u - C^*$ is a strictly positive and rational P^N-cycle over $K(u)$. Hence it determines a strictly positive and rational cycle Y of the product $P^N\times S$ over K, every component of which has the projection S on S and such that $Y(u)=V^*\cdot L_u - C^*$. Since S is of dimension $N-2$, our previous argument shows again that every component of Y has the projection V^* on V^*. Let Z be a component of Y, and let T' be the projection on S of a component of the intersection $Z\cap x^*\times S$. Since this intersection is nonempty, T' exists actually and it is a subvariety of S of dimension at least $N-3$ defined over K. From the same criterion of multiplicity one as before, we conclude that T' is contained in T. Since the dimension of T' is at least equal to the dimension of T, we get $T=T'$; hence in particular T is contained in $\mathfrak{E}_1$.

Now let $\bar{x}^*$ be a generic point of V^* over $k_0(x^*)$. Let x and $\bar{x}$ be the points of V corresponding to x^* and $\bar{x}^*$. We may assume that they have representatives $(1,x_1,\cdots,x_n)$ and $(1,\bar{x}_1,\cdots,\bar{x}_n)$ respectively. Let $X_0,X_1,\cdots,X_n$ be the letters to describe equations in P^n. Then m-fold products of the form

$$(X_i-x_iX_0)(X_j-x_jX_0)\cdots(X_k-x_kX_0)$$

for $1\leqq i\leqq j\leqq\cdots\leqq k\leqq n$ correspond to hyperplanes in P^N, and, by the criterion of multiplicity one, all of them contain the tangent plane of V^* at x^*. They determine a subspace of T. We may assume $\bar{x}_1\neq x_1$. Put $y_i=\bar{x}_i-x_i/\bar{x}_1-x_1$ for $i=1,\cdots,n$. We shall show that $k_0(x)(\bar{x})$ is separably algebraic over $k_0(x)(y)$. Otherwise, there exists at least one nontrivial derivation D of $k_0(x)(\bar{x})$ over $k_0(x)(y)$. Let $F_j(X)$ be a polynomial in $X_1,\cdots,X_n$ with coefficients in k_0 such that $F_j(\bar{x})=0$. We then get $y_iD\bar{x}_1-D\bar{x}_i=0$ $(2\leqq i\leqq n)$, $\sum_{i=1}^{n}\partial F_j/\partial\bar{x}_i\cdot D\bar{x}_i=0$. Therefore, the determinant of these n linear equations must be zero. Hence we get $\sum_{i=1}^{n}\partial F_j/\partial\bar{x}_i\cdot(x_i-\bar{x}_i)=0$ for every $F_j(X)$. However, since x is a generic point of V over $k_0(\bar{x})$, this shows that V is contained in, hence coincides with the tangent plane of V at $\bar{x}$. This is the case we have excluded in the beginning. Thus the linear system on V which is determined by T is not "composite with an algebraic pencil," hence its general member must be irreducible by the theorem of Bertini [21, 9]. This contradicts to the conclusion we have arrived before, i.e., to the fact that T is contained in $\mathfrak{E}_1$.

2. The following three lemmas hold also for an arbitrary algebraic surface V:

LEMMA 1. *Let x^* be a simple point of V^*. Then we can find a hyperplane L in P^N containing the tangent plane of V^* at x^* such that $V^* \cdot L$ consists of two nonsingular curves with distinct tangents locally at x^* for $m \geqq 2$.*

Proof. Let x be the point of V which corresponds to x^*. Let L^{n-1} and H^{n-1} be independent generic hyperplane and hypersurface of order $m-1$ over $k_0(x)$ passing both through x. Then $V \cdot L^{n-1}$ and $V \cdot H^{n-1}$ are irreducible curves at least locally at x which are transversal to each other at x on V. Let L' be the hyperplane in P^N which corresponds to $L^{n-1} + H^{n-1}$. Then $V^* \cdot L'$ corresponds biregularly to $V \cdot (L^{n-1} + H^{n-1})$. Therefore $V^* \cdot L'$ consists of two nonsingular curves with distinct tangents locally at x^*.

LEMMA 2. *Let x^* be a simple point of V^*. Then the tangent plane of V^* at x^* meets V^* only at x^* for $m \geqq 2$.*

Proof. We can use the same notations as in the above proof. Suppose that $\bar{x}^*$ is a point of V^* distinct from x^*. Let $\bar{x}$ be the corresponding point of V. If L^{n-1} and H^{n-1} are taken to be generic over $k_0(x, \bar{x})$, then L' does not pass through $\bar{x}^*$. Therefore the tangent plane of V^* at x^* can not contain $\bar{x}^*$.

LEMMA 3. *Let x^* and $\bar{x}^*$ be two distinct simple points of V^*. Then the tangent planes of V^* at x^* and at $\bar{x}^*$ do not intersect with each other for $m \geqq 3$.*

Proof. Let x and $\bar{x}$ be the points of V which correspond to x^* and $\bar{x}^*$. Let $H_1{}^{n-1}$ and $H_2{}^{n-1}$ be independent generic hypersurfaces of order $m-2$ over $k_0(x, \bar{x})$ passing both through x. Also let $L_1{}^{n-1}$, $L_1'{}^{n-1}$, $L_2{}^{n-1}$ and $L_2'{}^{n-1}$ be four hyperplanes passing through $\bar{x}$, but not through x. Let L_i be the hyperplanes in P^N which correspond to $L_i{}^{n-1} + L_i'{}^{n-1} + H_i{}^{n-1}$ for $i = 1, 2$. Then both L_1 and L_2 contain the tangent plane of V^* at $\bar{x}^*$, while $V^* \cdot L_1$ and $V^* \cdot L_2$ are transversal to each other at x^* on V^*. Therefore $L_1 \cdot L_2$ is a subspace of P^N of dimension $N-2$ which contains the tangent plane of V^* at $\bar{x}^*$ and which is transversal to V^* at x^*. The existence of such a subspace implies that the tangent planes of V^* at x^* and at $\bar{x}^*$ are disjoint.

3. We shall now assume that V has only "negligible singularities," i.e., that V does not contain any multiple curve. We can then add the following three supplements to Theorem 1:

SUPPLEMENT 1. *The set $\mathfrak{F}$ of hyperplanes in P^N, which are not trans-*

versal to V^* at one point of V^* at least, is a closed set over k_0 of dimension at most $N-1$ for $m \geqq 1$.

Proof. First of all the singular locus of V^* is a closed set over k_0 consisting of a finite number of points. The dual hyperplanes of these points form a closed set over k_0 in P'^N. On the other hand, let x^* be a generic point of V^* over k_0 and let L be a generic hyperplane of P^N over $k_0(x^*)$ containing the tangent plane of V^* at x^*. If u is the dual point of L, then u has a locus over k_0 of dimension at most $N-1$. Let L' be a specialization of L over k_0 and let x'^* be a specialization of x^* over this specialization. If x'^* is a multiple point of V^*, then L' is a member of the dual hyperplane of x'^*. If x'^* is a simple point of V^*, then L' contains the tangent plane of V^* at x'^*. This follows from the fact that the tangent plane of V^* at x'^* is the unique specialization of the tangent plane of V^* at x^* over the specialization $x^* \rightarrow x'^*$ with reference to k_0. Conversely let L' be a hyperplane in P^N containing a tangent plane of V^* at a simple point x'^* of V^*. Then L' is a specialization of L over k_0, as we can show by the following general argument: The totality of hyperplanes in P^N containing the tangent plane of V^* at x^* is a subspace T of P'^N of dimension $N-3$ defined over $k_0(x^*)$. Similarly a subspace T' of P'^N of dimension $N-3$ is attached to x'^*. Since the specialization T'' of T over the specialization $x^* \rightarrow x'^*$ with reference to k_0 is carried by T' and since T' and T'' are linear spaces of the same dimension, we have $T''=T'$. In particular a generic member of T' over $k_0(x'^*)$ is a specialization of L over the specialization $x^* \rightarrow x'^*$ with reference to k_0. Since L' is a member of T' and since T' is defined over $k_0(x'^*)$, we see that L' is a specialization of that generic member over $k_0(x'^*)$. Therefore L' is a specialization of L over the specialization $x^* \rightarrow x'^*$ with reference to k_0. Since a hyperplane L' of P^N is not transversal to V^* at a point x'^* of their intersection if and only if either x'^* is multiple on V^* or x'^* is simple on V^* and L' contains the tangent plane of V^* at x'^*, our assertion is proved.

SUPPLEMENT 2. *The set of hyperplanes in P^N, which are not transversal to V^* at two points of V^* at least, is contained in a closed set $\mathfrak{C}_2$ over k_0 of dimension at most $N-2$ for $m \geqq 3$.*

Proof. Since the set of singular points of V^* is closed over k_0, the set of hyperplanes in P^N containing at least two of them is a finite set of subspaces of dimension $N-2$ in P'^N, which is again closed over k_0. In the next place let a^* be one of the multiple points of V^*, and let x^* be a generic

point of V^* over k_0, hence also over $k_0(a^*)$. Then we can speak of a generic hyperplane L of P^N over $k_0(x^*, a^*)$ containing the tangent plane of V^* at x^* and passing through a^*. The dual point of L has a locus over $k_0(a^*)$, and the dimension of this locus is at most $N-2$ by Lemma 2. The union of such varieties for all a^* is closed over k_0. Moreover, if L' is a hyperplane in P^N containing a tangent plane of V^* at a simple point x'^* of V^* and passing through a multiple point a^* of V^*, then L' is a specialization of a generic L corresponding to the same a^* over the specialization $x^* \rightarrow x'^*$ with reference to $k_0(a^*)$. This can be proved exactly in the same way as in the proof of Supplement 1 using Lemma 2. Finally let x^* and $\bar{x}^*$ be independent generic points of V^* over k_0. Then we can speak of a generic hyperplane L of P^N over $k_0(x^*, \bar{x}^*)$ containing the tangent planes of V^* at x^* and at $\bar{x}^*$. The dual point of L has a locus over k_0, and the dimension of this locus is at most $N-2$ by Lemma 3. Moreover, if L' is a hyperplane in P^N containing tangent planes of V^* at two simple points x'^* and $\bar{x}'^*$, then L' is a specialization of L over the specialization $(x^*, \bar{x}^*) \rightarrow (x'^*, \bar{x}'^*)$ with reference to k_0. This can be proved in the same way as in the proof of Supplement 1 using Lemma 3. We can take as $\mathfrak{C}_2$ the union of the above three types of closed sets over k_0.

Our final supplement is more delicate. Let x^* be a simple point of V^*, and let C^* be a positive divisor of V^* passing through x^*. Then we can find an element f of the local ring of V^* at x^* such that its divisor (f) represents C^* locally at x^*. We shall assume that f is contained in the square, but not in the cube of the maximal ideal of the local ring. Then the residue class of f with respect to the cube of the maximal ideal is a quadratic form in the Zariski tangent space of V^* at x^*, and this quadratic form is uniquely determined up to a constant factor by C^*. If its discriminant is not zero, then x^* is called an *ordinary double point* of C^*.

SUPPLEMENT 3. *The set of hyperplanes in P^N, which intersect with V^* along curves with non-ordinary multiple points at simple points of V^*, is contained in a closed set $\mathfrak{C}_3$ over k_0 of dimension at most $N-2$ for $m \geqq 3$.*

Proof. Let x^* be a generic point of V^* over k_0, and let L be a generic hyperplane over $k_0(x^*)$ containing the tangent plane of V^* at x^*. Then x^* is the only point of V^* at which L is not transversal to V^*. Otherwise, from Lemmas 2, 3 we conclude that the dual point u of L is of dimension at most $N-4$ over $k_0(x^*)$. However, this dimension is actually $N-3$, and this is a contradiction. In particular u is not contained in any dual

hyperplane of a multiple point of V^*. Also the point x^* has no other specialization than itself over $k_0(u)$, hence x^* is purely inseparable over $k_0(u)$. Hence the locus U of u over k_0 is of dimension exactly equal to $N-1$. A nonempty subset of U is called to be open over k_0, if its complement is closed over k_0. Let U_0 be an affine representative of U in which u has a representative (u). We note that U_0 is an open set over k_0. Let U_1 be the subset of U which is obtained from U_0 by subtracting the singular locus, the union of dual hyperplanes of multiple points of V^* and the set $\mathfrak{E}_2$ which we introduced in Supplement 2. Then, by our previous remarks U_1 is nonempty, hence it is open over k_0. Moreover, if (u') is a point of U_1, the corresponding hyperplane in P^N is not transversal to V^* at exactly one simple point x'^* of V^*. Thus we have a single-valued map $(u')\rightarrow x'^*$ of U_1 into V^*. Let V_0^* be any, but fixed affine representative of V^*. Let (x^*) be the corresponding representative of x^*. We may assume that (x_1^*, x_2^*) are separating variables of $k_0(x^*)$ over k_0. Then we can find $N-2$ polynomials $F_j(X)$ in the defining ideal over k_0 of V_0^* such that $\det(\partial F_j/\partial x_i^*)_{i\neq 1,2}$ is not zero. Since x^* is purely inseparable over $k_0(u)$, a certain power of this determinant is an element $\phi(u)$ of $k_0(u)$. Here ϕ is a function on U_1 defined over k_0. In the same way we can associate to each x_i^* a function ϕ_i on U_1 defined over k_0. Let U_2 be the complement of U_1 of the poles of ϕ_i and the zeros of ϕ. Then U_2 is open over k_0. Moreover, if (u') is a point of U_2, the corresponding point x'^* on V^* has a representative in V_0^*, and (x_1^*, x_2^*) form local coordinates of V^* at x'^*. We shall denote by $\partial/\partial x_\alpha^*$ the derivations of $k_0(x^*)$ over $k_0(x_\beta^*)$ normalized by $\partial x_\alpha^*/\partial x_\alpha^*=1$ for $\alpha,\beta=1,2$ $(\alpha\neq\beta)$. Then $\det(\sum_i u_i\partial^2 x_i^*/\partial x_\alpha^*\partial x_\beta^*)$ is purely inseparable over $k_0(u)$. Therefore, its certain power determines a function ψ on U_2 over k_0. We note that ψ is regular on U_2 (cf. [8]), and ψ is not identically zero by Lemma 1. Therefore, if we denote by $\mathfrak{E}_3$ the union of the boundary of U_2 and the zeros of ψ in U_2, then $\mathfrak{E}_3$ is a closed set over k_0 of dimension at most $N-2$. It follows from the construction that $\mathfrak{E}_3$ satisfies our demands.

In the following we fix one m by $m\geqq 3$. If u is a point of the dual space P'^N of P^N, and if L_u is the corresponding hyperplane in P^N, then the biregular transform C_u on V of $V^*\cdot L_u$ is a member of $\mathfrak{L}_m$, and C_u is rational over $k_0(u)$. If we define $\mathfrak{E}$ as the union of $\mathfrak{E}_i$ for $i=1,2,3$, then we can summarize our results in the following way:

CONCLUSION. *There exist two closed sets $\mathfrak{F}$ and $\mathfrak{E}$ over k_0 in P'^N of dimensions at most $N-1$ and $N-2$ respectively such that* (i) *if u is not*

contained in $\mathfrak{F}$, *the corresponding* C_u *is an irreducible*[4] *nonsingular curve*; (ii) *if* u' *is a point of* $\mathfrak{F}-\mathfrak{E}$, *then* $C_{u'}$ *is still irreducible and has one and only one multiple point*; (iii) *if this multiple point is simple on* V, *then it is an ordinary double point of* C_u.

Now, consider the Grassmann variety G of straight lines in P'^N. The variety G has a generic point over k_0, and this point generates a purely transcendental extension over k_0. Moreover the totality of straight lines in P'^N, which either lie on $\mathfrak{F}$ or intersect with $\mathfrak{E}$, is a closed subset of G over k_0 of co-dimension at least equal to one. On the other hand to each straight line D in P'^N corresponds a subspace L of P^N of dimension $N-2$. If D is generic over k_0, the intersection product $V^* \cdot L$ is defined and consists of, say β, distinct generic points of V^* over k_0. If we associate the Chow point of $V^* \cdot L$ to D, we get a rational map over k_0 of G into the β-fold symmetric product $V^*(\beta)$ of V^*. The points of $V^*(\beta)$, which correspond to β non-distinct points of V^*, form a closed set over k_0 of co-dimension one. The inverse image of this closed set in G is again a closed set over k_0 of co-dimension one. We take an algebraic point over k_0 in G which is contained neither in this set nor in the closed set we introduced before. If k_0 is an infinite field, even a rational point over k_0 can be found under the above condition. It is an open question whether we can do the same thing, by taking a larger m if necessary, when k_0 is finite. Let D be the straight line in P'^N which corresponds to the point under consideration. Then we get a linear pencil $\{C_u\}$ with the parameter straight line D such that the parametrization, i.e., the correspondence $u \rightarrow C_u$ is defined over the algebraic closure k of k_0. Since D does not intersect with $\mathfrak{E}$, (i) *every* C_u *is irreducible.* Since D is not contained in $\mathfrak{F}$, (ii) *there is a finite number of points, say* $a_1, \cdots, a_\alpha$, *on* D *such that* C_u *become singular only at these points.* (iii) *Each* C_{a_i} *has one and only one multiple point, and if this multiple point is simple on* V, *it is an ordinary double point of* C_{a_i} *for* $i=1, \cdots, \alpha$. Also since we have avoided another kind of exceptional straight lines, by the criterion of multiplicity one, (iv) *the base points* $A_1, \cdots, A_\beta$ *of the pencil are simple on* V, *and any two members of the pencil are transversal to each other on* V *at* $A_1, \cdots, A_\beta$. A linear pencil $\{C_u\}$ with the above four properties will be called a *general linear pencil* on V.

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[4] It is clear that C_u is a disjoint sum of nonsingular curves. However C_u must be connected by the principle of degeneration. We can avoid this deep principle simply by taking the union of $\mathfrak{F}$ and $\mathfrak{E}$ as our $\mathfrak{F}$.

2. Construction of generalized Jacobian varieties by Chow's method.

4. Let C be an arbitrary irreducible curve in a projective space P^n and let $\mathfrak{o}$ be the intersection of local rings of C at its multiple points. Then $\mathfrak{o}$ determines C uniquely up to biregular transformations. We shall denote by K_0 the smallest field of definition of C. Also, if $\mathfrak{q}$ is a positive 0-cycle in P^n, its Chow point will be denoted by $(\mathfrak{q})$. If K is a field, we write $K(\mathfrak{q})$ instead of $K((\mathfrak{q}))$. On the other hand, a divisor of C means always a divisor of C in the sense of Weil, i.e., every component of a divisor shall be a simple point of C. If f is a function on C, we shall consider its *divisor in the sense of valuation theory*. This is not a good terminology; it means the regular image on C of the divisor of the function induced by f on the normalization of C.

Now, if $\mathfrak{p}$ is an arbitrary divisor of C and if $\mathfrak{q}$ is a 0-cycle in P^n carried by C, we write $\mathfrak{p} \rightarrow \mathfrak{q}$ whenever there exists an element f of $\mathfrak{o}$ such that $\mathfrak{q} - \mathfrak{p}$ is the divisor of f in the sense of valuation theory [14]. Here f does not vanish at any multiple point of C if and only if $\mathfrak{q}$ is also a divisor of C. Therefore the relation $\mathfrak{p} \rightarrow \mathfrak{q}$ can be reversed if and only if $\mathfrak{q}$ is a divisor of C. In other words, if we restrict to divisors, the arrow relation is an equivalence relation. If $\mathfrak{p}$ is a divisor of C, we shall denote by $|\mathfrak{p}|$ the set of all positive 0-cycles $\{\mathfrak{q}\}$ in P^n satisfying $\mathfrak{p} \rightarrow \mathfrak{q}$, and we call $|\mathfrak{p}|$ the *complete linear system on C determined by $\mathfrak{p}$*. We note that every member of $|\mathfrak{p}|$ has the same degree, and we call it the degree of $|\mathfrak{p}|$. Here we must show that $|\mathfrak{p}|$ is actually a linear system on C. This can be done by using the following lemma due to Zariski [20]: *Let V be an arbitrary variety in P^n, and let $t_0, \cdots, t_n$ be the coordinates functions of the representative cone of V. Let h be a homogeneous element of degree m in the function field of the cone such that h/t_i^m is regular on V outside the closed set defined by $t_i = 0$ for each i. Then h can be expressed as a polynomial in the t_i with coefficients in $\mathfrak{K}$ provided m is not less than a fixed integer independent of h.* Now, let $\mathfrak{L}_m$ be the linear system on C which we defined before. If we take m sufficiently large, we can find a positive divisor $\mathfrak{r}$ of C such that $\mathfrak{p} + \mathfrak{r}$ is an intersection product of C with a hypersurface H_0 of order m. We shall show that $|\mathfrak{p}|$ coincides with $\mathfrak{L}_m - \mathfrak{r}$. Let $\mathfrak{q}$ be an arbitrary member of $|\mathfrak{p}|$. Then, by definition we can find an element f of $\mathfrak{o}$ such that $\mathfrak{q} - \mathfrak{p}$ is the divisor of f in the sense of valuation theory. In other words we can find two hypersurfaces of the same order F and F_0 such that F_0 does not pass through any multiple point of C and $(F - F_0) \cdot C = \mathfrak{q} - \mathfrak{p}$ holds. If h is the function on the representative cone of C corresponding up to a constant to $F + H_0 - F_0$,

then, by Zariski's lemma there exists a hypersurface H of order m such that $(F+H_0-F_0)\cdot C=\mathfrak{q}+\mathfrak{r}=H\cdot C$ holds. Therefore $\mathfrak{q}$ is a member of $\mathfrak{L}_m-\mathfrak{r}$. The converse is obvious. We note also that if $\mathfrak{p}$ is positive, we have only to take m not less than the maximum of the degree of $\mathfrak{p}$ and of the fixed integer in Zariski's lemma for $V=C$.

Now, if d is a positive integer, we shall denote by $C(d)$ the d-fold symmetric product of C. Also we shall denote by $C_0(d)$ the Chow variety of positive divisors on C of degree d. It is clear that $C_0(d)$ is an open subvariety of $C(d)$, and $C(d)$ lies in a projective space P^t of dimension $t=C_n{}^{d+n}-1$. The following lemma is proved, not exactly in this form, by Chow [3]:

LEMMA 1. *Every point of $C_0(d)$ is simple on $C(d)$. If W is a subvariety of $C(d)$ of dimension r, then the order of W is at least equal to* $\operatorname{ord}(C)^r$. *The extremal value* $\operatorname{ord}(C)^r$ *is attained if W is the variety of a "sufficiently general" linear system on C and also if $r=d$.*

Here a linear system $\mathfrak{L}$ is sufficiently general if the following condition is satisfied: Take a field K over which the corresponding subvariety of $C(d)$ is defined. Let $\mathfrak{p}=\sum_{i=1}^{d}P_i$ be a generic member of $\mathfrak{L}$ over K. Then all the d points $P_1,\cdots,P_d$ are generic points of the curve C over K and distinct from each other; moreover, any r of these points are independent with respect to each other over K and determine the remaining $d-r$ points uniquely.

On the other hand, it is better to state here Chow's theorem on fibre systems in a form suited for our purpose. First of all, we must recall the definition of fibre systems. Let V and W be irreducible, but possibly "incomplete varieties" in projective spaces, and let p be a function on V with values in W. We assume that p is regular on V and the point-set theoretical inverse F_y of each point y of W is an irreducible variety of the same dimension and of the same order. We then call $\{F_y\}$ a *fibre system* on V over W. If we take the set of Chow points $\{y^*\}$ of the fibres F_y, we get an irreducible, possibly incomplete variety W^* such that W is a one-to-one rational transform of W^*. This variety W^* is called the *associated variety* of the fibre system. According to a theorem of Chow [2], *a point y^* is simple on W^* if and only if the corresponding fibre F_y is simple on V.*

5. Now we shall construct the so-called generalized Jacobian variety of C as an associated variety of a certain fibre system on $C_0(d)$. The following lemma is well known if C is nonsingular:

LEMMA 2. *If* $\mathfrak{a}$ *is a rational divisor of* C *over a field* K *and if* $\dim|\mathfrak{a}| \geqq 0$, *then* $|\mathfrak{a}|$ *contains a rational cycle over* K. *Also if* $\mathfrak{a}$ *is of degree* d, *the subvariety of* $C(d)$ *which corresponds to* $|\mathfrak{a}|$ *is defined over* K.

Proof. By assumption there exists at least one positive 0-cycle $\mathfrak{b}$ in P^n satisfying $\mathfrak{a} \rightarrow \mathfrak{b}$. Let f be the element of $\mathfrak{o}$ such that $\mathfrak{b} - \mathfrak{a}$ is the divisor of f in the sense of valuation theory. If (f) is the usual divisor of f, we have $(f) + \mathfrak{a} \succ 0$. Since $\mathfrak{a}$ is rational over K, by a theorem of Weil [16, p. 239] f can be expressed as $f = \sum_{\alpha} c_\alpha f_\alpha$, where the c_α are linearly independent elements of $\mathfrak{K}$ over K and the f_α are functions on C defined over K satisfying $(f_\alpha) + \mathfrak{a} \succ 0$. Moreover, since the singular locus of C is a closed set over K, by the same theorem the f_α are elements of $\mathfrak{o}$. In other words, we can find an element $f' = f_\alpha$ of $\mathfrak{o}$ which is defined over K and which satisfies $(f') + \mathfrak{a} \succ 0$. If $\mathfrak{b}'$ is the sum of $\mathfrak{a}$ and the divisor of f' in the sense of valuation theory, $\mathfrak{b}'$ is a member of $|\mathfrak{a}|$. Moreover $\mathfrak{b}'$ is rational over K, and this proves the first part. The above reasoning shows also that we can find a set of linearly independent functions f_α all defined over K and forming a base over $\mathfrak{K}$ of the vector space over $\mathfrak{K}$ of elements f in $\mathfrak{o}$ satisfying $(f) + \mathfrak{a} \succ 0$. The number of f_α is then necessarily finite. Let (x) be a set of independent variables over K, and let $\mathfrak{x}$ be the sum of $\mathfrak{a}$ and the divisor of $\sum_{\alpha} x_\alpha f_\alpha$ in the sense of valuation theory. Then $|\mathfrak{a}|$ coincides with the totality of specializations of $\mathfrak{x}$ over K. However since $\mathfrak{x}$, hence also its Chow point $(\mathfrak{x})$ is rational over the regular extension $K(x)$ of K, we see that $(\mathfrak{x})$ has a locus over K. It is clear that this locus is the subvariety of $C(d)$ which corresponds to $|\mathfrak{a}|$. This proves the second part.

Now, let g be the *arithmetic genus* of C. We fix an integer d by $d > 2g - 2$, and we shall imitate the method of Chow [3] to construct the generalized Jacobian variety of C. For that purpose we shall first verify the following assertion: Let $\mathfrak{m}$ be a positive divisor of C of degree d the components of which are independent generic points of C over K_0. Then the complete linear system $|\mathfrak{m}|$ is sufficiently general in the sense stated next to Lemma 1. In fact, if $\mathfrak{M}$ is the subvariety of $C(d)$ which corresponds to $|\mathfrak{m}|$, then, by Lemma 2, the Chow points z of $\mathfrak{M}$ is rational over $K_0(\mathfrak{m})$. In other words $K_0(z)$ is contained in $K_0(\mathfrak{m})$. However, since $K_0(\mathfrak{m})$ is of maximal dimension d over K_0, we conclude that $(\mathfrak{m})$ is a generic point of $\mathfrak{M}$ over $K_0(z)$. In particular, the linear system $|\mathfrak{m}|$ is sufficiently general. As a consequence of this, and of Lemma 1, the order of $\mathfrak{M}$ is equal to $\operatorname{ord}(C)^r$. Moreover every specialization of $\mathfrak{M}$ over K_0 is a subvariety of $C(d)$.

On the other hand, we must also verify that any specialization of a complete linear system is again a complete linear system. We remark here that our complete linear systems are defined by stricter equivalence relation than the usual one. We take an integer m not less than d and not less than the fixed integer in Zariski's lemma such that $\dim \mathfrak{L}_m = \operatorname{ord}(C) \cdot m - g$. According to the Riemann-Roch theorem for C [14], if $\mathfrak{p}$ is a divisor of C of degree d, we have $\dim |\mathfrak{p}| = d - g$. Assume that $\mathfrak{p}$ is positive, and let $\mathfrak{P}$ be the subvariety of $C(d)$ which corresponds to $|\mathfrak{p}|$. Then $\mathfrak{P}$ is the unique specialization of $\mathfrak{M}$ over the specialization $\mathfrak{m} \to \mathfrak{p}$ with reference to K_0. This is the precise formulation of what we have stated above. Let $\mathfrak{M}'$ be a specialization of $\mathfrak{M}$ over the specialization $\mathfrak{m} \to \mathfrak{p}$ with reference to K_0. If we project each point of $\mathfrak{m}$ and $\mathfrak{p}$ from the same generic subspace of dimension $n-2$ in P^n over $K = K_0(\mathfrak{M}, \mathfrak{M}', \mathfrak{m}, \mathfrak{p})$, thus obtaining two hypersurfaces of order d passing through $\mathfrak{m}$ and $\mathfrak{p}$, and if we add to them another generic hypersurface of order $m - d$ which is independent over K to the above subspace of P^n, we get two hypersurfaces H and H' of order m in P^n. It follows from the construction that H' is a specialization of H over the specialization $(\mathfrak{M}, \mathfrak{m}) \to (\mathfrak{M}', \mathfrak{p})$ with reference to K_0. We put $\mathfrak{r} = C \cdot H - \mathfrak{m}$ and $\mathfrak{r}' = C \cdot H' - \mathfrak{p}$. It is then clear that $\mathfrak{r}$ and $\mathfrak{r}'$ are positive divisors on C such that $|\mathfrak{m}| = \mathfrak{L}_m - \mathfrak{r}$ and $|\mathfrak{p}| = \mathfrak{L}_m - \mathfrak{r}'$. Since $\mathfrak{M}'$ is a specialization of $\mathfrak{M}$ over the specialization $\mathfrak{r} \to \mathfrak{r}'$ with reference to K_0, it is carried by $\mathfrak{P}$. However, since $\mathfrak{M}'$ is irreducible and of the same dimension as $\mathfrak{P}$, we get $\mathfrak{M}' = \mathfrak{P}$.

From now on, Chow's argument [3] can be taken over verbatim to the present case. Hence we are satisfied with outlining his method by referring to the original paper of Chow for details of the proof: Let J be the locus of the Chow point z of $\mathfrak{M}$ over K_0. Then the set of Chow points of the varieties $\mathfrak{P}$ of complete linear systems on C of degree d is a subset J_0 of J. Moreover, if $(\mathfrak{p})$ is a point of $C_0(d)$ and if $\Psi(\mathfrak{p})$ denotes the Chow point of the variety $\mathfrak{P}$ which corresponds to $|\mathfrak{p}|$, then Ψ is a function on $C_0(d)$ having K_0 as a field of definition. This follows from Lemma 2. Also Ψ is "continuous" in the sense that it is commutative with specializations. If we remember the nonsingular character of $C_0(d)$, we can conclude that Ψ is regular on $C_0(d)$.

Finally, we shall show that J_0 is an open subvariety of J over K_0. Since the singular locus of C is a closed set over K_0, we conclude that $C(d) - C_0(d)$ is a closed set over K_0. We note also that every component of $C(d) - C_0(d)$ is biregularly equivalent to $C(d-1)$. In particular such a component is of

dimension $d-1$, hence we can attach a positive $(d-1)$-cycle X in P^t which is rational over K_0 and which has the same components as $C(d)-C_0(d)$. Now let $\mathfrak{P}$ be a subvariety of $C(d)$ which corresponds to a point of J. Then, by a theorem in the theory of associated forms [1], the condition that $\mathfrak{P}$ is contained in $C(d)-C_0(d)$ can be expressed by a set of doubly homogeneous equations in the Chow coordinates of $\mathfrak{P}$ and the Chow coordinates of X with coefficients in the prime field, and this gives a set of homogeneous equations in the Chow coordinates of $\mathfrak{P}$ with coefficients in K_0. This set of equations defines a closed subset of J over K_0, and J_0 is its complement.

In conclusion, the totality of $\mathfrak{P}\cap C_0(d)$ forms a fibre system on $C_0(d)$ with J_0 as its associated variety. Since $C_0(d)$ is nonsingular, Chow's theorem on fibre systems guarantees the *nonsingular character* of J_0.

6. Now, we can modify d under the condition $d>2g-2$ so that C carries a rational divisor $\mathfrak{a}$ of degree d over K_0. If θ is a divisor on C of degree zero, we define $\Phi(\theta)$ to be the Chow point of the variety which corresponds to $|\theta+\mathfrak{a}|$. It then follows from Lemma 2 that $\Phi(\theta)$ is rational over $K_0(\theta)$. Moreover, every point of J_0 can be written as $\Phi(\theta)$ with some θ. Since the set of divisors on C of degree zero forms a group, we can introduce an abstract group structure in J_0 so that Φ becomes a homomorphism. Then the law of composition in J_0 is "continuous" in the sense that it is commutative with specializations. Also, by Lemma 2 law is normal over K_0 in the sense of Weil [17, p. 51]. Therefore, from the nonsingular character of J_0 we can conclude that the "naive group operation" in J_0 is an actual group operation. In other words J_0 turns out to be a commutative group variety over K_0. Here we have $\Phi(\theta)=0$ if and only if $\theta\rightarrow 0$. The group of all such θ is a homomorphic image of the group of units in $\mathfrak{o}$, the kernel being the multiplicative group of $\mathfrak{K}$. Therefore Φ gives a rational homomorphism over K_0 [3] of the divisor group on C of degree zero onto J_0 such that the kernel is the group of principal divisors on C in the above sense. Thus J_0 and Φ are uniquely determined by C up to isomorphic transformations σ: $(J_0,\Phi)\rightarrow(\sigma J_0,\sigma\circ\Phi)$. Here, is is better to remark that J_0 is determined together with its projective embedding by the curve C in P^n and the integer d, while the group structure in J_0 as well as the homomorphism Φ depend also on the choice of the reference divisor $\mathfrak{a}$ of C. If we do not want to define the group structure *over* K_0, we can take any divisor of degree d as a reference divisor.

We shall show finally that J_0 is the generalized Jacobian variety of C in the sense of Rosenlicht. Let K be an extension of K_0 over which C carries a

rational divisor A of degree one. We can then define over K the canonical function ϕ of C by $\phi(M)=\Phi(M-A)$ for any simple point M of C. Let $M_1,\cdots,M_g$ be independent generic points of C over K and put $z=\sum_{i=1}^{g}\phi(M_i)$. Also, let $\mathfrak{m}^*=\sum_{i=1}^{d}M_i^*$ be a generic member of $|\sum_{i=1}^{g}M_i-g\cdot A+\mathfrak{a}|$ over $F=K(\sum_{i=1}^{g}M_i)$. Then, first of all, z is rational over F. Also, $\sum_{i=1}^{g}M_i$ determines a complete linear system of dimension zero [15]. Therefore F is a subfield of $K(\mathfrak{m}^*)$ by Lemma 2, and F is purely inseparable over $K(z)$. Since $K(\mathfrak{m}^*)$ is regular over $K(z)$, we conclude $F=K(z)$. This shows that our group variety J_0 is isomorphic to the generalized Jacobian variety of C with $\mathfrak{o}$ as its reference semi-local ring [15]. We state some of our results in the following way:

THEOREM 2. *Let C be an arbitrary irreducible curve in P^n having K_0 as the smallest field of definition. Then we can construct its generalized Jacobian variety J_0 over K_0 in a projective space P^N. The construction depends on a positive integer d, but once d is fixed, it is unique.*

In the Appendix we shall determine the "linear equivalence class" of the hyperplane sections of J_0, thus revealing the nature of the construction.

7. We shall treat more in detail the case when C has one and only one ordinary double point. Let K be the smallest algebraically closed field of definition for C and for the ambient surface. If C^* is the normalization of C over K, then C^* is nonsingular. We shall show that the genus γ of C^* is equal to $g-1$. Let $\mathfrak{o}$ be the local ring of C at the double point, say Q. Since Q is an ordinary double point, the completion $\mathfrak{o}^*$ of $\mathfrak{o}$ consists of pairs of formal power series with the same constant terms. Therefore, if $\mathfrak{O}$ is the integral closure of $\mathfrak{o}$, its completion $\mathfrak{O}^*$ with respect to the semi-local topology is a direct sum of two formal power series rings. In particular, the factor module $\mathfrak{O}^*/\mathfrak{o}^*$ is a vector space of dimension one over $\mathfrak{K}$. Since the vector space $\mathfrak{O}/\mathfrak{o}$ is canonically isomorphic to $\mathfrak{O}^*/\mathfrak{o}^*$, the space $\mathfrak{O}/\mathfrak{o}$ is also of dimension one. However this dimension is equal to $g-\gamma$ [14], hence $g=\gamma+1$. The above argument implies also that Q corresponds to two distinct points Q_1^* and Q_2^* of C^*. Moreover, a function f on C^* which is regular at Q_1^* and Q_2^* belongs to $\mathfrak{o}$ if and only if $f(Q_1^*)=f(Q_2^*)$ holds. This fact will be used presently.

Now, we shall denote by $C(d-1,Q)$ the difference $C(d)-C_0(d)$. Also, for $\alpha=1,2$, we shall denote by $C^*(d-1,Q_\alpha^*)$ the set of Chow points of

positive divisors on C^* of degree d containing Q_α^* as a component. It is clear that $C(d-1, Q)$ is biregularly equivalent to $C(d-1)$. Similarly $C^*(d-1, Q_\alpha^*)$ is biregularly equivalent to $C^*(d-1)$. Moreover, $C^*(d)$ is nonsingular and it is the normalization of $C(d)$ over K.

Let $M_1, \cdots, M_d$ be independent generic points of C over K, and put $\mathfrak{m} = \sum_{i=1}^{d} M_i$. Let $\mathfrak{m}^* = \sum_{i=1}^{d} M_i^*$ be the unique transform of $\mathfrak{m}$ on C^*. Then, the complete linear system $|\mathfrak{m}|$ is transformed to a linear subsystem of $|\mathfrak{m}^*|$. Let $\mathfrak{M}^{**}$ be the subvariety of $C^*(d)$ of dimension $d-g$ which corresponds to this linear subsystem. Also, let $\mathfrak{M}^*$ be the subvariety of $C^*(d)$ which corresponds to $|\mathfrak{m}^*|$. Then, the Chow point z^* of $\mathfrak{M}^*$ has a locus J^* over K, and J^* is the Jacobian variety of C^* [3]. We shall show that the locus J^{**} of the Chow point z^{**} of $\mathfrak{M}^{**}$ is birationally equivalent to J over K. Let $\mathfrak{M}$ be the subvariety of $C(d)$ which corresponds to $|\mathfrak{m}|$, and let z be the Chow point of $\mathfrak{M}$. Then z is purely inseparable over $K(z^{**})$ and also z^{**} is purely inseparable over $K(z)$. Since $K(\mathfrak{m}^*) = K(\mathfrak{m})$ is regular over $K(z^{**})$ and over $K(z)$, we get $K(z^{**}) = K(z)$. This proves our assertion. In particular, we can apply Lemma 1 to $\mathfrak{M}^{**}$, and $\mathfrak{M}^{**}$ is of order equal to $\mathrm{ord}(C^*)^{d-g}$. Moreover, every specialization of $\mathfrak{M}^{**}$ over K is a subvariety of $C^*(d)$.

Now, let $(\mathfrak{P}, \mathfrak{P}^{**})$ be an arbitrary specialization of $(\mathfrak{M}, \mathfrak{M}^{**})$ over K. Then, this is the unique extension of $\mathfrak{M}^{**} \to \mathfrak{P}^{**}$ over K, and also of $\mathfrak{M} \to \mathfrak{P}$ over K if $\mathfrak{P}$ is not contained in $C(d-1, Q)$. This follows from the fact that $C^*(d)$ is the normalization of $C(d)$ over K. We shall examine the case when $\mathfrak{P}$ is contained in $C(d-1, Q)$. Since linear equivalence is preserved under specializations, $\mathfrak{P}^{**}$ is contained in a variety $\mathfrak{P}^*$ of a complete linear system of degree d on C^*, say $\mathfrak{G}$. Since $d > 2\gamma$, we have $\dim \mathfrak{G} = d - \gamma = d - g + 1$. Moreover $\mathfrak{P}^* \cap C^*(d-1, Q_\alpha^*)$ are the varieties corresponding to $(\mathfrak{G} - Q_\alpha^*) + Q_\alpha^*$ for $\alpha = 1, 2$. Since $d - 1 > 2\gamma - 1$, these linear systems are both of dimension $d - g$. Since the variety $\mathfrak{P}^{**}$ is contained in the union of $C^*(d-1, Q_\alpha^*)$, it must coincide with one of the $\mathfrak{P}^* \cap C^*(d-1, Q_\alpha^*)$. We note that these two varieties are distinct, because $\mathfrak{G} - Q_\alpha^*$ have no fixed points. We shall now show that the totality of $\mathfrak{P}^{**}$ induces a fibre system on $C^*(d) - \bigcap_\alpha C^*(d-1, Q_\alpha^*)$ with J^{**} as its associated variety. We shall first show that the fibres have no common point. Let $(\mathfrak{q}^*)$ be a point of $C^*(d)$ not in $\bigcap_\alpha C^*(d-1, Q_\alpha^*)$. If $\mathfrak{q}^*$ is free from Q_1^* and Q_2^*, there is no ambiguity. Suppose that $\mathfrak{q}^*$ contains, say Q_1^*. Let $\mathfrak{P}^{**}$ be a fibre passing through $(\mathfrak{q}^*)$. Then $\mathfrak{P}^{**}$ must be contained in

$C^*(d-1, Q_1^*)$. Otherwise, since $\mathfrak{P}^{**}$ can not be contained in $C^*(d-1, Q_2^*)$, the corresponding subvariety $\mathfrak{P}$ of $C(d)$ will be a variety of a complete linear system on C. Thus by a remark we made in the beginning of this section, $\mathfrak{q}^*$ must contain Q_2^*, which is not the case. We conclude that $\mathfrak{P}^{**}$ is the variety of $(|\mathfrak{q}^*| - Q_1^*) + Q_1^*$, and this shows that in any case $(\mathfrak{q}^*)$ determines the fibre $\mathfrak{P}^{**}$ uniquely. The correspondence $(\mathfrak{q}^*) \to \mathfrak{P}^{**}$ defines a single-valued map from $C^*(d) - \bigcap_\alpha C^*(d-1, Q_\alpha^*)$ onto J^{**}. Since we know that this map is a function on $C^*(d)$ with K as a field of definition, and since $C^*(d)$ is nonsingular, it is regular on $C^*(d) - \bigcap_\alpha C^*(d-1, Q_\alpha^*)$. This completes our proof.

Since $C^*(d)$ is nonsingular, by Chow's theorem J^{**} is nonsingular. From this and from what we remarked before, it follows that the birational correspondence between J and J^{**} is regular on J^{**} and biregular on J_0. On the other hand, there is a function p on J^{**} defined over K with values in J^*. In fact, using our previous notation $K(\mathfrak{m}^*)$ is regular over $K(z^{**})$, while z^* is rational over $K(\mathfrak{m}^*)$ and purely inseparable over $K(z^{**})$. Hence p can be defined over K by $p(z^{**}) = z^*$. Since p is single-valued on J^{**} and since J^{**} is nonsingular, p is regular on J^{**}. For $\alpha = 1, 2$, the Chow points of varieties corresponding to linear systems of the form $(\mathfrak{G} - Q_\alpha^*) + Q_\alpha^*$ form a subvariety J_α^*. Since J_α^* is the associated variety of the fibre system of the varieties of complete linear systems on $C^*(d-1, Q_\alpha^*)$, it is biregularly equivalent to J^*. Actually p induces such a biregular map on J_α^*. We note that J_1^* and J_2^* are disjoint. Now, by a similar reasoning we conclude that $J - J_0$ is biregularly equivalent to J^*. Actually, the birational correspondence between J and J^{**} induces such a biregular map on J_α^*. Thereby, if z_α^* are points of J_α^* for $\alpha = 1, 2$, the images of z_1^* and z_2^* on $J - J_0$ coincide if and only if $p(z_1^*) - p(z_2^*) = \phi^*(Q_1^*) - \phi^*(Q_2^*)$ holds. Here ϕ^* is the canonical function on C^*. We summarize some of our results in the following way:

SUPPLEMENT. *If C has one and only one ordinary double point, the normalization J^{**} of J is nonsingular. Moreover, if J^* is the Jacobian variety of the nonsingular model C^* of C, then $J - J_0$ is biregularly equivalent to J^* and is a "double variety" of J.*

Also, we can see easily that $p^{-1}(z^*)$ is biregularly equivalent over $K(z^*)$ to a projective straight line for every z^* on J^*. We can derive from this that J^{**} is the projective line bundle over J^* which is obtained by "completing" the affine line bundle $J_0 = J^{**} - J_1^* - J_2^*$ over J^*. Moreover the

invariant of this bundle in the sense of Weil [18] is given by the class of linear equivalence of $\Theta^*_{\phi^*(Q_1^*)} - \Theta^*_{\phi^*(Q_2^*)}$. Here Θ^* is the divisor of J^* which is defined as the transform on J^* of the $(\gamma-1)$-fold product of C^* by ϕ^*.

3. Néron varieties of a nonsingular surface.

8. Let C and C' be two irreducible curves in P^n such that C' is a specialization of C over a field K. Let J and J' be the completed generalized Jacobian varieties of C and C' respectively. If C and C' have the same arithmetic genus, say g, and if we use the same reference integer d greater than $2g-2$ in the constructions of J and J', then J and J' have the same ambient space P^N and have the same dimension. In a separate paper we proved the following lemma [5]:

LEMMA. *If $\mathfrak{L}_m$ and $\mathfrak{L}'_m$ refer to C and C' for a sufficiently large m, then a specialization of a member of $\mathfrak{L}_m$ over the specialization $C \to C'$ with reference to K is a member of $\mathfrak{L}'_m$.*

We can now prove the following "compatibility theorem."

THEOREM 3. *If C is nonsingular, and if C' is either nonsingular or has one and only one ordinary double point, then J' is the unique specialization of J over the specialization $C \to C'$ with reference to K.*

Proof. Let J'' be a specialization of J over the specialization $C \to C'$ with reference to K. Our purpose is to show that $J'' = J'$. Thereby we can replace (J, C) by its generic specializations over K and in this way we can extend K arbitrarily. Therefore we can assume from the beginning that J'' and C' are both defined over K. Let F be the smallest field of definition of C containing K. Let Γ be the graph of the function Ψ on $C(d)$ with values in J which we introduced in 5; similarly Γ' is defined for C'. We shall show that Γ' is the unique specialization of Γ over the specialization $(J, C) \to (J'', C')$ with reference to K.

Let Γ'' be a specialization of Γ over this specialization. Let L be a linear subspace of co-dimension d in the ambient space of Γ which is generic over an algebraically closed field of definition E of Γ'' containing F. Then, any point ξ' in $L \cdot \Gamma''$ is a generic point of some component of Γ'' over E. Moreover, in this way we get a generic point of every component of Γ'' over E. Let ξ be a point in $L \cdot \Gamma$ which is specialized to ξ'. Since ξ is a generic point of Γ over F, it is of the form $(\mathfrak{m}) \times z$. Here $(\mathfrak{m})$ is a generic point of $C(d)$

over F, and z is the Chow point of the variety $\mathfrak{M}$ associated with $|\mathfrak{m}|$. On the other hand, since the specialization of $C(d)$ over the specialization $C \to C'$ with reference to K is carried by $C'(d)$, and since $C(d)$ and $C'(d)$ have the same order, $C'(d)$ is the unique specialization of $C(d)$ over that specialization. Therefore Γ'' is carried by the product $C'(d) \times P^N$. In particular, we can write ξ' in the form $(\mathfrak{m}') \times z'$, where $(\mathfrak{m}')$ is a point of $C'(d)$ and z' is the Chow point of the specialization $\mathfrak{M}'$ of $\mathfrak{M}$ over the specialization $\xi \to \xi'$ with reference to K. Since $\mathfrak{M}'$ is carried by $C'(d)$, we conclude from Lemma 1, Section 2 that $\mathfrak{M}'$ is an irreducible variety. Since z' and $(\mathfrak{m}')$ are points of J'' and $\mathfrak{M}'$ respectively, we have $\dim_E z' \leqq g$ and $\dim_{E(z')}(\mathfrak{m}') \leqq d - g$. However, since $(\mathfrak{m}') \times z'$ is of dimension d over E, the inequality signs can not hold in the above relations. We shall show that ξ' is a point of Γ'. Assume first that $\mathfrak{m}'$ is not a divisor of C'. Then, it contains the double point Q of C' at least once. Since $(\mathfrak{m}')$ is a generic point of $\mathfrak{M}'$ over $E(z')$, we conclude that $\mathfrak{M}'$ is contained in $C'(d-1, Q) = C'(d) - C'_0(d)$. If C^* is the nonsingular model of C', there are only a finite number of positive divisors $\mathfrak{m}_\alpha^*$ on C^* having $\mathfrak{m}'$ as a projection. Let U_α be the projections on $C'(d)$ of the varieties corresponding to the complete linear systems $|\mathfrak{m}_\alpha^*|$ on C^*. Then $\mathfrak{M}'$ is contained in $\bigcup_\alpha U_\alpha \cap C'(d-1, Q)$ according to the previous lemma. However, this is a closed set over $K(\mathfrak{m}')$ of dimension $d-g$. Hence $\mathfrak{M}'$ must coincide with one of its components. In particular, the Chow point z' of $\mathfrak{M}'$ must be algebraic over $K(\mathfrak{m}')$, hence a fortiori over $E(\mathfrak{m}')$. Thus $(\mathfrak{m}')$ is a generic point of $C'(d)$ over E, and this is a contradiction. Therefore $\mathfrak{m}'$ must be a divisor of C'. In this case, as in 5, we can find two positive divisors $\mathfrak{r}$ and $\mathfrak{r}'$ of C and C' respectively such that $\mathfrak{r}'$ is a specialization of $\mathfrak{r}$ over the specialization $(\mathfrak{m}, \mathfrak{M}) \to (\mathfrak{m}', \mathfrak{M}')$ with reference to K and such that $|\mathfrak{m}| = \mathfrak{L}_m - \mathfrak{r}$ and $|\mathfrak{m}'| = \mathfrak{L}'_m - \mathfrak{r}'$ for a sufficiently large m. Since we know that $\mathfrak{M}'$ is irreducible, we can conclude from the previous lemma that $\mathfrak{M}'$ is the variety associated with $|\mathfrak{m}'|$. Therefore ξ' is a point of Γ'. Since ξ' is a generic point over E of an arbitrary component of Γ'', we conclude that Γ'' is an integer multiple of Γ'. However, since $\mathrm{pr}_1(\Gamma'')$ coincides with $\mathrm{pr}_1(\Gamma') = C'(d)$ as a specialization of $\mathrm{pr}_1(\Gamma) = C(d)$, we get $\Gamma'' = \Gamma'$.

Finally, let $R = (M_1, \cdots, M_{d-g})$ be a set of $d-g$ independent generic points of C over F, and let R' be an isolated specialization of R over the specialization $(J, C) \to (J'', C')$ with reference to K. Then R' is a set of $d-g$ independent generic points of C' over K. If we define W to be the subvariety of $C(d)$ whose points correspond to divisors of C containing the

$d-g$ points of R as components, W is biregularly equivalent to $C(g)$. We define W' similarly for C' and R'. It is clear that W' is the unique specialization of W over the specialization $(R, J, C) \rightarrow (R', J'', C')$ with reference to K. On the other hand, the intersection product $\Gamma \cdot (W \times P^N)$ is defined and is a birational correspondence between W and J. We shall show that the intersection product $\Gamma' \cdot (W' \times P^N)$ is also defined and is a birational correspondence between W' and J'. Let $(\mathfrak{m}') \times z'$ be a generic point of some component of the intersection $\Gamma' \cap W' \times P^N$ over the algebraic closure F of $K(R')$. Then, by the results in 7 we get $\dim_{F(\mathfrak{m}')} z' \leqq 1$ in all cases. Here the equality holds if and only if $\mathfrak{m}'$ contains the double point at least twice. Since $(\mathfrak{m}') \times z'$ is of dimension g over F, we conclude that the components of $\mathfrak{m}'$ are R' and g independent generic points of C' over F. Our assertion follows from this. Therefore, $J' = \mathrm{pr}_2[\Gamma' \cdot (W' \times P^N)]$ is a specialization of $J = \mathrm{pr}_2[\Gamma \cdot (W \times P^N)]$ over the specialization $J \rightarrow J''$ with reference to K, whence $J' = J''$. q. e. d.

9. From now on, we fix a nonsingular algebraic surface V in P^n. Let $\{C_u\}$ be a general linear pencil on V whose parametrization is defined over an algebraically closed field k. We may assume, if necessary, that k is the algebraic closure of the smallest field of definition of V. Since V is nonsingular, it is normal and hence all C_u have the same arithmetic genus, say g [22]. Also, if A is one of the base points, k being algebraically closed, A is simple on V and is rational over k. Now, let J_u be the completed generalized Jacobian variety of C_u constructed with reference to a fixed integer d greater than $2g-2$. Then J_u is defined over $k(u)$, hence there exists a subvariety $\mathcal{J}$ of the product $D \times P^N$ with k as a field of definition such that $\mathcal{J} \cdot (u \times P^N) = u \times J_u$ for all generic points u of D over k. Then, by the compatibility theorem the same formula remains valid for all u on D. If we apply the criterion of multipilicity one to this situation [16, p. 141], we can conclude that a point $u \times z$ of $\mathcal{J}$ is simple on $\mathcal{J}$ as long as z is simple on J_u. If we define a function p on $\mathcal{J}$ over k by $p(u \times z) = u$, then p is regular on $\mathcal{J}$. Therefore $\{u \times J_u\}$ is a fibre system on $\mathcal{J}$ over D. Since D is nonsingular, it is biregularly equivalent to the associated variety of the fibre system. We call $\mathcal{J}$ the *Néron variety of V associated with $\{C_u\}$*, and we state our main theorem the first part of which is trivial:

THEOREM 4. *To every general linear pencil on a nonsingular surface we can associate its Néron variety. It is the variety of the parametrization of Jacobian varieties of the members of the pencil. The singular locus of the*

Néron variety is contained in the union of singular loci of degenerate fibres. In particular, it has only negligible singularities.

Now, we can use $d \cdot A$ as the reference divisor to introduce a group structure in $(J_u)_0$ over $k(u)$. At the same time we can normalize the canonical function ϕ_u of C_u by $\phi_u(A) = 0$. Under these agreements, we can state the following assertions: (i) *If (x,y) is a pair of points of J_u which is specialized to (x',y') over the specialization $u \rightarrow u'$ with reference to k, and if both x' and y' are points of $(J_{u'})_0$, then $x'+y'$ is the unique specialization of $x+y$ over the above specialization.* (ii) *If Γ_u and $\Gamma_{u'}$ are the graphs of the normalized canonical functions on C_u and $C_{u'}$, then $\Gamma_{u'}$ is the unique specialization of Γ_u over the specialization $u \rightarrow u'$ with reference to k.*

Let M be a generic point of V over k, and let C_u be the member of the pencil passing through M. Then u is rational over $k(M)$. Moreover, if we put $x = \phi_u(M)$, we have $k(u)(M) = k(u)(x)$, i.e., $k(M) = k(u,x)$. In other words, we can define a birational map ϕ over k from V into $\mathcal{J}$ by $\phi(M) = u \times \phi_u(M)$. We note that the image of ϕ is not contained in the singular locus of $\mathcal{J}$.

10. We shall now treat the connection of Albanese varieties and linear differential forms of the first kind attached to V and $\mathcal{J}$. Here, the Albanese variety of an arbitrary variety U is a pair of an Abelian variety A and a function f on U with values in A satisfying the two conditions: (i) The image of f generates A. (ii) If h is a function on U with values in an Abelian variety B, there exists a homomorphism α from A into B such that $h = \alpha \circ f$ + constant. These two properties determine (A,f) up to isomorphic transformations, and the existence of (A,f) is proved by Chow and Matsusaka [4, 10]. On the other hand, we summarized some of the basic properties of differential forms of the first kind already elsewhere [6]. They are mainly due to Koizumi [8]. We need also a result of Kawahara [7], which, in an apparently stronger form, can be stated as follows: *Let f be a rational map of a variety V into another variety U such that the image variety is not contained in the singular locus of U. Then the associated linear map δf maps every differential form of the first kind on U to a differential form of the first kind on V.* We shall use the terminology such as injective, surjective and bijective instead of "one-to-one into," "onto" and "one-to-one onto." The mappings we consider later are linear mappings of differential forms.

We shall first consider the symmetric product $V(d)$ of an arbitrary variety V in P^n. We must get an information about the singular locus of

$V(d)$. The following lemma is a consequence of Chow's theorem [2], but we shall give a direct proof:

LEMMA 1. *If $P_1,\cdots,P_d$ are distinct simple points of V, the Chow point of $\sum_{i=1}^{d} P_i$ is simple on $V(d)$.*

Proof. Let (a_i) be the homogeneous coordinates of P_i, and let x_i be d independent generic points of V over an algebraically closed field of definition K of V over which P_i are rational. Let b and y be the Chow points of $\sum_{i=1}^{d} P_i$ and $\sum_{i=1}^{d} x_i$ respectively. Since any linear coordinates transformation in P^n corresponds to a special type of linear coordinates transformation in the ambient space of $V(d)$, we can assume the following: We can normalize (a_i) as $a_{i0}=1$. If (x_i) is the representative of x_i satisfying $x_{i0}=1$, then $(x_{i1},\cdots,x_{ir})$ form a set of local coordinates of V at any P_k. If we put $D_j(X)=\prod_{d\geqq k>i}(X_{kj}-X_{ij})$, we have $D_j(a)\neq 0$ for $j=1,\cdots,r$.

Now, let (y) be the representative of y which corresponds to the above representation. Let $(y_{1j},\cdots,y_{dj})$ be the elementary symmetric functions of $(x_{1j},\cdots,x_{dj})$. Then, all the y_{ij} appear as components of (y). Let $G_{ij}(X,Y)=0$ be the set of equations for (x_{ij}) and (y_{ij}) with coefficients in the prime field. Then, $\det(\partial G_{ij}/\partial X_{kl})$ coincides with $\prod_{j=1}^{r} D_j(X)$ up to a possible change of sign. Therefore, this determinant is different from zero at $(X_{ij})=(a_{ij})$. Since $(x_{i1},\cdots,x_{ir})$ form a set of local coordinates of V at any P_k, we can find $n-r$ polynomials $F_\alpha(X)$ in the defining ideal of V over K such that $\det(\partial F_\alpha/\partial X_\beta)$ for $\beta=r+1,\cdots,n$ are different from zero at $(X)=(a_k)$ for $k=1,\cdots,d$. Therefore, we can apply a criterion of multiplicity one [16, p. 66] to conclude the following: Let $(i_1,\cdots,i_d)$ be any permutation of $(1,\cdots,d)$. Then, the specialization $(x_1,\cdots,x_d)\rightarrow(a_{i_1},\cdots,a_{i_d})$ is of multiplicity one over the specialization $(y_{ij})\rightarrow(b_{ij})$ with reference to K. Therefore, by a stronger reason the specialization $(y)\rightarrow(b)$ is also of multiplicity one over the specialization $(y_{ij})\rightarrow(b_{ij})$ with reference to K. Hence, by another criterion of multiplicity one [16, pp. 127, 139], b is simple on $V(d)$.

Let $P_2,\cdots,P_d$ be $d-1$ distinct simple points of V, and let P_1 be an arbitrary point of V. Then, the Chow point of $\sum_{i=1}^{d} P_i$ can be considered as the value of a function Ψ on V at P_1. The previous lemma implies that the

image of Ψ is not contained in the singular locus of $V(d)$. Therefore, any rational map of $V(d)$ into an Abelian variety is regular along the image of Ψ [17, p. 27]. In particular, if (A,F) is the Albanese variety of $V(d)$, then the composite function $F\circ\Psi$ is defined.

LEMMA 2. *The Albanese varieties of V and $V(d)$ are isomorphic. More precisely, $(A,F\circ\Psi)$ is the Albanese variety of V. Also, $\delta\Psi$ is injective as a mapping of linear differential forms of the first kind.*

Proof. The first part can be proved easily, hence we can omit its proof. Let V_d be the d-fold product $V\times\cdot\cdot\cdot\times V$ of V. Let θ be a linear differential form of the first kind on $V(d)$. If p is the natural projection of V_d onto $V(d)$, then $\delta p\cdot\theta$ is a linear differential form of the first kind on V_d. Therefore, if we denote by p_i the projection of V_d to its i-th factor, we have $\delta p\cdot\theta=\sum_{i=1}^{d}\delta p_i\cdot\omega_i$ with linear differential forms of the first kind ω_i on V [8]. Since $\delta p\cdot\theta$ is invariant with respect to the interchange of factors of V_d, we have $\delta p_i\cdot\omega_i=\delta p_i\cdot\omega_j$ for any i and j. Therefore, we have $\omega_i=\omega_j$ for any i and j, whence we can drop the suffices. It is clear that $\omega=\delta\Psi\cdot\theta$ holds. Since p is separable, $\theta\neq 0$ implies $\delta p\cdot\theta\neq 0$, whence $\omega\neq 0$. In other words, $\delta\Psi$ is injective.

On the other hand, the following lemma can be proved in the same way as Lemma 2. It is in fact a corollary of Lemma 2.

LEMMA 3. *Let (J,f) be the Albanese variety of a nonsingular curve C. Then δf is bijective as a mapping of linear differential forms of the first kind.*

11. We can now treat the connetcion between our nonsingular surface V and the Néron variety $\mathcal{G}$, i.e., we can prove the following theorem:

THEOREM 5. *Let (A,f) be the Albanese variety of $\mathcal{G}$. Then $(A,f\circ\phi)$ is the Albanese variety of V. Moreover, $\delta\phi$ is injective as a mapping of linear differential forms of the first kind.*

Proof. Let u be a generic point of D over k, and let $M_1,\cdot\cdot\cdot,M_g$ be independent generic points of C_u over $k(u)$. Put $\mathfrak{m}=\sum_{i=1}^{g}M_i$ and $x=\sum_{i=1}^{g}\phi_u(M_i)$. Then we have $k(\mathfrak{m})=k(u,x)$. Therefore, a function ψ on $\mathcal{G}$ with values in $V(g)$ is defined over k such that the Chow point $(\mathfrak{m})$ of $\mathfrak{m}$ is the value of ψ at $u\times x$. We note that the image of ψ is not contained in the singular locus of $V(g)$. Now, in the statement of the theorem $f\circ\phi$ is defined since f

is regular along the image of ϕ. Let (B,h) be the Albanese variety of V. Then, there exists a homomorphism β from B into A such that $f\circ\phi=\beta\circ h$ $+$ constant. On the other hand, let $M_1,\cdots,M_g$ be distinct simple points of V. Then, we can define a function H on $V(g)$ with values in B such that $\sum_{i=1}^{g} h(M_i)$ is the value of H at the Chow point of $\sum_{i=1}^{g} M_i$. The function H is regular along the image of ψ, hence $H\circ\psi$ is defined. We can then find a homomorphism α from A into B such that $H\circ\psi=\alpha\circ f+$ constant. After these preparations, take an extension K of k over which A, B, f and h are all defined. As before, let u be a generic point of D over K, and let $M_1,\cdots,M_g$ be independent generic points of C_u over $K(u)$. Then we have

$H\circ\psi(u\times\sum_{i=1}^{g}\phi_u(M_i))=\sum_{i=1}^{g}h(M_i)$, and also

$$H\circ\psi(u\times\sum_{i=1}^{g}\phi_u(M_i))=\alpha\circ f(u\times\sum_{i=1}^{g}\phi_u(M_i))+\text{constant}.$$

However, since $\alpha\circ f$ induces a homomorphism of $u\times J_u$ into B up to a possible translation in B [17, p. 34], we have

$$\alpha\circ f(u\times\sum_{i=1}^{g}\phi_u(M_i))=\sum_{i=1}^{g}\alpha\circ f(u\times\phi_u(M_i))+\text{constant}.$$

Therefore, we get $\sum_{i=1}^{g}h(M_i)=\sum_{i=1}^{g}\alpha\circ f\circ\phi(M_i)+$ constant. Since h and $\alpha\circ f\circ\phi$ are both regular at the base points of $\{C_u\}$, by specializing $M_2,\cdots,M_g$ to one of these base points we get $h=\alpha\circ f\circ\phi+$ constant. Therefore,

$$\alpha\circ\beta\circ h=\alpha\circ f\circ\phi+\text{constant}=h+\text{constant},$$

hence $\alpha\circ\beta=1$. In the same way, if we put $x=\sum_{i=1}^{g}\phi_u(M_i)$, we have

$$\begin{aligned}\beta\circ H\circ\psi(u\times x)&=\beta(\sum_{i=1}^{g}h(M_i))=\sum_{i=1}^{g}\beta\circ h(M_i)=\sum_{i=1}^{g}f\circ\phi(M_i)+\text{constant}\\&=f(u\times\sum_{i=1}^{g}\phi_u(M_i))+\text{constant}=f(u\times x)+\text{constant},\end{aligned}$$

i. e., we have $\beta\circ H\circ\psi=f+$ constant. Therefore,

$$\beta\circ\alpha\circ f=\beta\circ H\circ\psi+\text{constant}=f+\text{constant},$$

hence $\beta\circ\alpha=1$. This completes the proof of the first part.

Next we shall show that $\delta\phi$ is injective. Let ϖ be a linear differential

form of the first kind on $\mathcal{J}$ such that $\delta\phi \cdot \varpi = 0$. Let K be an extension of k over which ϖ is defined, and let u be a generic point of D over K. Assume, for a moment, that $\mathrm{Tr}_{u \times J_u} \varpi \neq 0$. Then, by Lemma 3 we have

$$0 = \mathrm{Tr}_{C_u}(\delta\phi \cdot \varpi) = \delta(\mathrm{Tr}_{C_u}\phi) \cdot \varpi \neq 0,$$

and this is a contradiction. Therefore, we have $\mathrm{Tr}_{u \times J_u} \varpi = 0$. We may assume that u is the value of a numerical function on $\mathcal{J}$ with k as a field of definition. Let x be a generic point of J_u over $K(u)$, and let $(v_1, \cdots, v_g)$ be a set of independent variables in $K(u, x)$ over $K(u)$ such that $K(u, x)$ is separably algebraic over $K(u, v)$. Since ϖ is defined over K, we can write $\varpi(u \times x)$ in the form $fdu + \sum_{i=1}^{g} h_i dv_i$ with f and h_i in $K(u, x)$. Since $\mathrm{Tr}_{u \times J_u} \varpi = 0$, we have $fDu + \sum_{i=1}^{g} h_i Dv_i = 0$ for any derivation D in $K(u, x)$ over $K(u)$. Therefore, h_i are all zero and we get $\varpi(u \times x) = fdu$. However, u or $1/u$ can be included in a set of local coordinates on $\mathcal{J}$ everywhere outside the singular loci of degenerate fibres. Therefore, if $\varpi \neq 0$, we get $(\varpi) = (f) - 2 \cdot J_\infty$. Since ϖ is of the first kind, we have $(\varpi) \succ 0$. Thus, a strictly negative divisor is linearly equivalent to a positive divisor on the variety $\mathcal{J}$ with negligible singularities! But this is a contradiction. Therefore $\varpi = 0$, hence $\delta\phi$ is injective.

Appendix.

In this Appendix we shall analyze Chow's construction of Jacobian varieties. The restriction to Jacobian varieties is merely a matter of simplicity. Let C be a nonsingular curve of genus g in P^n with K_0 as the smallest field of definition. Let J be the Jacobian variety of C which is constructed in P^N by Chow's method with respect to an integer d greater than $2g-2$. Our purpose is to determine the linear equivalence class of hyperplane sections of J. As before, we introduce a group structure in J with reference to a divisor $\mathfrak{a}$ on C of degree d. Let K be an extension of K_0 over which $\mathfrak{a}$ is rational and the canonical function ϕ of C is defined. If $P_1, \cdots, P_{g-1}$ are $g-1$ points of C, the set of points of J of the form $\sum_{i=1}^{g-1} \phi(P_i)$ is a subvariety Θ of J of dimension $g-1$ with K as a field of definition. If z is a point of J, then Θ_z is the image of Θ under the translation $x \rightarrow x + z$ of J. On the other hand if $\sum_i x_i$ is a hyperplane section of C, then, by Abel's theorem [17, p. 39] $c = \phi(\sum_i x_i) = \sum_i \phi(x_i)$ is independent of the choice of the hyperplane.

In particular, c is rational over K. We shall now prove the following assertion:

THEOREM. *A hyperplane section of J is linearly equivalent to* $\Theta_z - \Theta + \mathrm{ord}(C)^{d-g+1}\cdot\Theta$ *with*

$$z = (d-g+1)\mathrm{ord}(C)^{d-g}\,c - \mathrm{ord}(C)^{d-g+1}\phi(\mathfrak{a}).$$

Proof. Let (X), (Y) and (Z) be the sets of letters to describe equations in P^n, P^t and P^N respectively. Let (u) be a set of $n+1$ quantities. In general, if w is a point of a projective space, its homogeneous coordinates will be denoted by (w). Let $x^1,\cdots,x^d$ be d points in P^n, and let y be the Chow point of $\sum_{i=1}^{d} x^i$. Then, we have a relation of the form

$$\prod_{i=1}^{d}\Big(\sum_{j=0}^{n} u_j x^i{}_j\Big) = \sum_{j=0}^{t}\omega_j(u)y_j.$$

Here $\omega_j(u)$ are monomials of degree d in $u_0,\cdots,u_n$. Now, let L_v be a subspace of P^t of dimension $t-r$ which is defined by $\sum_{j=0}^{t} v^i{}_j Y_j = 0$ for $i=1,\cdots,r$. Let W^r be a positive cycle in P^t of order s such that the intersection product $W\cdot L_v$ is defined. Let $\sum_{\beta=1}^{s} y^\beta$ be this 0-cycle in P^t. Also, let (v^0) be a set of $t+1$ quantities. Then, if z is the Chow point of W, we have the following relation

$$\prod_{\beta=1}^{s}\Big(\sum_{j=0}^{t} v^0{}_j y^\beta{}_j\Big) = \sum_{j=0}^{N}\Omega_j(v^0,\cdots,v^r)z_j.$$

Here $\Omega_j(v^0,\cdots,v^r)$ are monomials of degree s in each $v^\alpha{}_0,\cdots,v^\alpha{}_t$ for $\alpha=0,\cdots,r$. After these preliminary remarks, we put $r=d-g$ and $s=\mathrm{ord}(C)^{d-g}$. Let $\sum_{j=0}^{n} u^\alpha{}_j X_j = 0$ be $r+1$ independent generic linear equations over K for $\alpha=0,\cdots,r$. They define $r+1$ hyperplanes in P^n intersecting C at $\sum_{i=1}^{d} x^{\alpha i}$. Let H be the hyperplane in P^N with the equation

$$\sum_{j=0}^{N}\Omega_j(\omega(u^0),\cdots,\omega(u^r))Z_j = 0.$$

We shall determine the intersection product $J\cdot H$ of J and H. We denote by $L_{\omega(u)}$ the subspace of P^t of dimension $t-r$ which is defined by $\sum_{j=0}^{t}\omega_j(u^\alpha)Y_j = 0$ for $\alpha=1,\cdots,r$. If $J\cdot H$ is not defined, J is contained

in H. We shall derive a contradiction from this assumption. Let z be a generic point of J over $K(u^0,\cdots,u^r)$, and let $\mathfrak{P}$ be the corresponding subvariety of $C(d)$. Then, the intersection product $\mathfrak{P}\cdot L_{\omega(u)}$ is defined. Let it be $\sum_{\beta=1}^{s} y^\beta$. Since z is a point of H, we have $\prod_{\beta=1}^{s}(\sum_{j=0}^{t}\omega_j(u^0)y^\beta{}_j)=0$, hence $\sum_{j=0}^{t}\omega_j(u^0)y_j=0$ for some $y=y^\beta$. However, since y is a point of $L_{\omega(u)}$, we also have $\sum_{j=0}^{t}\omega_j(u^\alpha)y_j=0$ for $\alpha=1,\cdots,r$. If $\sum_{j=1}^{d}x^i$ is the 0-cycle in P^n corresponding to y, we may assume that $\sum_{j=0}^{n}u^\alpha{}_jx^\alpha{}_j=0$ for $\alpha=1,\cdots,r$ and also $\sum_{j=0}^{n}u^0{}_jx^{r+1}{}_j=0$. In particular, $x^1,\cdots,x^{r+1}$ are independent generic points of C over $K(z)$, while $K(x^1,\cdots,x^d)$ is of dimension r over $K(z)$. This is a contradiction. Therefore $J\cdot H$ is defined. Now, let z be a generic point of some component W of $J\cdot H$ over the algebraic closure F of $K(u^0,\cdots,u^r)$. Then z is a generic point of J over $K_1=K(u^1,\cdots,u^r)$. Otherwise, (u^0) would be a set of independent variables over $K_1(z)$, which is not the case. Therefore, if $\mathfrak{P}$ is the subvariety of $C(d)$ which corresponds to z, then $\mathfrak{P}\cdot L_{\omega(u)}$ is again defined. Let it be $\sum_{\beta=1}^{s}y^\beta$. Then, some $y=y^\beta$ corresponds to a 0-cycle $\sum_{i=0}^{d}x^i$ such that $\sum_{j=0}^{n}u^\alpha{}_jx^\alpha{}_j=0$ for $\alpha=1,\cdots,r$ and $\sum_{j=0}^{n}u^0{}_jx^{r+1}{}_j=0$. We note that $x^1,\cdots,x^{r+1}$ are rational over F, and z is rational over $F(y)$. Therefore $x^{r+2},\cdots,x^d$ are independent generic points of C over F. Since W is the locus of the point z over F, it is of the form $\Theta_{z_\gamma-\phi(\mathfrak{a})}$ with $z_\gamma=\sum_{i=1}^{r+1}\phi(x^i)$. Conversely, if z_γ is as above, $\Theta_{z_\gamma-\phi(\mathfrak{a})}$ is obviously a component of $J\cdot H$. We shall now show that H is transversal to J along $\Theta_{z_\gamma-\phi(\mathfrak{a})}$, i.e., at z. We know that $\sum_{j=0}^{N}\Omega_j(\omega(u^0),\cdots,\omega(u^r))z_j=0$ is the only relation between $(u^0),\cdots,(u^r)$ with coefficients in $K(z)$. Also, if we put $r+1$ sets of indeterminates $(U^0),\cdots,(U^r)$ instead of $(u^0),\cdots,(u^r)$, then $\sum_{j=0}^{N}\Omega_j(\omega(U^0),\cdots,\omega(U^r))z_j$ is irreducible over $K(z)$. In fact, we have

$$\sum_{j=0}^{N}\Omega_j(\omega(U^0),\omega(u^1),\cdots,\omega(u^r))z_j=\prod_{\beta=1}^{s}\prod_{i=1}^{d}(\sum_{j=1}^{n}U^0{}_jx^{\beta i}{}_j).$$

Here $(x^{\beta 1},\cdots,x^{\beta d})$ for a complete set of conjugates over $K_1(y^\beta)=K_1(z)(y^\beta)$ for each β, and $(y^1,\cdots,y^s)$ form a complete set of conjugates over $K_1(z)$, whence our assertion. Suppose now that H is not transversal to J at z. Let (z) be normalized by $z_0=1$. Then, we can find $N-g$ polynomials

$F_\lambda(Z)$ in the defining ideal of J over K_0 such that the matrices $(\partial F_\lambda/\partial z_j)$ and

$$\begin{pmatrix} \partial F_\lambda/\partial z_j \\ \Omega_j(\omega(u^0),\cdots,\omega(u^r)) \end{pmatrix}$$

have the same rank $N-g$. We thus get a relation of the form

$$\sum_{j>0} A_j(z)\Omega_j(\omega(u^0),\cdots,\omega(u^r))=0$$

with $A_j(z)$ in $K_0(z)$. However since $\sum_{j=0}^{N} z_j\Omega_j(\omega(U^0),\cdots,\omega(U^r))$ is irreducible over $K(z)$, this must divide $\sum_{j>0} A_j(z)\Omega_j(\omega(U^0),\cdots,\omega(U^r))$. On the other hand, by taking a suitable coordinates transformation in P^n if necessary, we may assume that $\Omega_0(\omega(U^0),\cdots,\omega(U^r))$ is equal to $(U^0{}_0\cdots U^r{}_r)^{d_s}$. We are thus led to a contradiction. We have thus shown that $J\cdot H$ is of the form $\sum_\gamma \Theta_{z_\gamma-\phi(\mathfrak{a})}$, where the summation is extended over distinct $\Theta_{z_\gamma-\phi(\mathfrak{a})}$. However, since the correspondence $z\to\Theta_z$ is one-to-one [17, p. 76], the summation is extended over the distinct z_γ. Here, two distinct sets $(x^1,\cdots,x^{r+1})$ and $(x'^1,\cdots,x'^{r+1})$ correspond to different z. Otherwise we get $g=0$, and in this case everything is trivial. Thus, the number of γ is equal to $\operatorname{ord}(C)^{r+1}$, whence

$$\sum_\gamma \Theta_{z_\gamma-\phi(\mathfrak{a})}\sim\Theta_{\sum_\gamma z_\gamma-\sum_\gamma\phi(\mathfrak{a})}+(\operatorname{ord}(C)^{r+1}-1)\cdot\Theta$$

holds [17, p. 106]. It is now a simple matter to verify that

$$\sum_\gamma z_\gamma-\sum_\gamma\phi(\mathfrak{a})=(d-g+1)\operatorname{ord}(C)^{d-g}c-\operatorname{ord}(C)^{d-g+1}\phi(\mathfrak{a}).$$

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FULLY REDUCIBLE SUBGROUPS OF ALGEBRAIC GROUPS.*

By G. D. Mostow.[1]

Section 1. Introduction.

In the theory of Lie groups one often encounters statements which can be asserted for *connected* groups on the strength of the appropriate analogue for Lie algebras, but which may be invalid for non-connected groups. A good example of this phenomenon is the theorem of Sophus Lie that a solvable linear Lie algebra (over a field of characteristic 0) and hence a *connected* solvable linear Lie group can be simultaneously triangularized (upon extending the ground of its algebraic closure). This powerful principle is not valid for general solvable linear groups. In fact, an examination of proofs of results that are valid for *only* connected Lie groups often reveals that Lie's theorem has been used in an essential way.

On the other hand, there are some results which are deduced for connected groups from their Lie algebras which continue to ring true when the hypothesis of connectedness is dropped. In this paper we obtain several results of such a type. The results are related for most part to properties of fully reducible groups of linear transformations. Our central result concerns a decomposition of algebraic groups which is closely related to the Wedderburn decomposition of an associative algebra into the semi-direct sum of a semi-simple subalgebra and the radical.

Our decomposition for algebraic groups can be viewed as a result on group extensions. Any algebraic group is a finite extension of the connected component of its identity G_0. To what extent is the extension splittable? Put in another way, how large a normal *connected* subgroup N can we find so that N admits a complementary subgroup? The answer in any particular case depends on the arithmetic properties of the ground field. Nevertheless for a general ground field of characteristic zero, something can still be asserted.

THEOREM. *Let N be the set of unipotent elements (i.e. eigenvalues are*

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all 1) *of the radical of the algebraic group G. Let M be any maximal fully reducible subgroup of G. Then N is a connected normal subgroup and*

$$G = M \cdot N (\textit{semi-direct}).$$

For a general ground field this choice of N is the best possible.

THEOREM. *Any two maximal fully reducible subgroups of an algebraic group are conjugate under an inner automorphism.*

There is of course a similar result for algebraic Lie algebras. As an application we prove the following

THEOREM. *Let G be an algebraic group and let $\mathfrak{E}$ denote its enveloping associative algebra. Then there is a Wedderburn decomposition $\mathfrak{S} + \mathfrak{T}$ for $\mathfrak{E}$ ($\mathfrak{S}$ semi-simple, $\mathfrak{T}$ the radical) such that*

a) *$\mathfrak{S} \cap G$ is a maximal fully reducible subgroup of G;*

b) *$(I + \mathfrak{T}) \cap G$ is the subgroup of unipotent elements in the radical of G, I being the identity;*

c) *$\mathfrak{T} \cap \mathfrak{G}$ is the ideal of nilpotent elements in the radical of $\mathfrak{G}$, the Lie algebra of G.*

There is an analogous result for algebraic Lie algebras. We prove also

THEOREM. *A fully reducible group of automorphisms of a Lie algebra keeps a maximal semi-simple subalgebra invariant.*

THEOREM. *A fully reducible group of automorphisms of a solvable Lie algebra keeps a Cartan subalgebra invariant.*

In addition, there are analogous theorems for automorphisms of Lie groups. Some of the intermediate results have independent interest. For example (cf. Proposition 3.2), any rational representation of an algebraic group carries fully reducible subgroups into fully reducible subgroups. As a consequence, it can be asserted that the First Main Theorem of the Theory of Invariants is valid for fully reducible groups.

Another consequence of this same principle is the following

THEOREM. *A fully reducible group of automorphisms of a Lie or associative or Jordan algebra keeps a maximal semi-simple subalgebra invariant* (Section 5).

In conclusion, it may be noted that fully reducible subgroups behave

very much like compact subgroups of Lie groups. Indeed there is a very close relation between these two notions and we describe the relation elsewhere.

Section 2. Preliminaries.

2. 1. Terminology. We deal throughout with a ground field K of characteristic 0. We employ the Zariski topology for subsets of finite dimensional linear spaces, a closed set being by definition an algebraic manifold. The terms closure, relatively open, and connected are defined in terms of closed sets in the usual way. By an endomorphism of a linear space V is meant a linear map of V into itself. By a linear group we mean a group of automorphisms of a finite dimensional linear space. The linear space of endomorphisms of a linear space V is denoted by $E(V)$.

An *algebraic group* is a group of automorphisms of a linear space V which is relatively open in its closure in $E(V)$; or, what is the same, it is a group constituting the totality of automorphisms in some algebraic manifold in $E(V)$ (cf. [2]). By the *algebraic group hull* of a set S of automorphisms is meant the intersection of all algebraic groups containing S. It is the same as the closure of S in the set of automorphisms of V, if S is a group. The connected component of the identity G_0 of an algebraic group has as its associated ideal (of polynomial functions on $E(V)$ which vanish on G_0) a prime ideal and is a normal relatively open subgroup of finite index. It should be noted that if K is the field of real numbers, an algebraic group connected in the Zariski topology need not be connected in the euclidean topology; however for K the field of complex numbers, the two notions agree.

An endomorphism T of a linear space V defines by right translation a left invariant infinitesimal transformation T^* on $E(V)$; via the derivation $P(X) \rightarrow dP(X + sXT)/ds$ at $s = 0$ of the ring of polynomial functions $P(X)$ defined on $E(V)$. The *Lie algebra* of an algebraic group G is the totality of elements T in $E(V)$ such that T^* keeps invariant the ideal of polynomial functions vanishing on G (cf. [2]). An *algebraic Lie algebra* is by definition the Lie algebra of an algebraic group. An algebraic group and its connected component of the identity have the same Lie algebra. Algebraic Lie algebras are related to connected algebraic groups in much the same way as in the classical case real Lie algebras are related to connected Lie groups. In particular, a connected algebraic group keeps a linear subspace invariant if and only if its Lie algebra does; it is therefore fully reducible if and only if its Lie algebra is.

If T is an endomorphism of a linear space V, we denote by $V_b(T)$ the

totality of elements of V annihilated by some power of $T - bI$ where b is in K and I denotes the identity. We call $V_0(T)$ the nilspace of T and call T nilpotent if $V_0(T) = V$. If F is a family of endomorphisms of V, we mean by $V_0(F)$, the nilspace of F, the intersection of the nilspaces of the elements of F. For any endomorphism T there exist a unique fully reducible endomorphism s and a unique nilpotent endomorphism n such that $sn = ns$ and $s + n = T$. This decomposition is called "the Jordan sum decomposition of T," s and n being called the fully reducible and nilpotent parts of T respectively.

If T is an automorphism of the linear space V, then T has a unique product decomposition $T = s \cdot u$ where $s \cdot u = u \cdot s$, s is the fully reducible part of T and u is *unipotent*, i.e. $V_1(u) = V$. Any algebraic group contains the Jordan product components of each of its elements, and correspondingly every algebraic Lie algebraic contains the Jordan sum components of its elements. Algebraic Lie algebras are thus splittable in the sense of Malcev (cf. [7]).

Let F be a family of endomorphisms of a linear space V, and let $G(V)$ denote the group of automorphisms of V. By a *similarity* of F is meant the restriction to F of a mapping $f \rightarrow xfx^{-1}$ where x is in $G(V)$ and $xFx^{-1} = F$. A group G of similarities of F is called pre-fully reducible if there exists a fully reducible subgroup G_1 in $G(V)$ with G the restrictions to F of the similarities from G_1.

Let F be a family of endomorphisms of a linear space V which keeps a linear subspace W invariant. We denote by F_W (resp. $F_{V/W}$) the induced family of transformations of W (resp. V/W) and call these the W-part and V/W part of F respectively. If ρ is a representation of a Lie algebra $\mathfrak{G}$, we call $\mathfrak{G}$ ρ-reductive if $\rho(\mathfrak{G})$ is a fully reducible set. We denote by ad the adjoint representation of a Lie algebra, $\operatorname{ad} x(y)$ being by definition $[x, y]$.

2. 2. Cartan subalgebras. Let $\mathfrak{G}$ be a Lie algebra, let $\mathfrak{G}^0 = \mathfrak{G}$, let $\mathfrak{G}^n = [\mathfrak{G}^{n-1}, \mathfrak{G}]$, and let $\mathfrak{G}^\infty = \bigcap_n \mathfrak{G}^n$. $\mathfrak{G}$ is called nilpotent if $\mathfrak{G}^\infty = (0)$. By Engel's Theorem, $\mathfrak{G}$ is nilpotent if $\operatorname{ad} x$ is nilpotent for all x in $\mathfrak{G}$. A Cartan subalgebra of a Lie algebra $\mathfrak{G}$ is any nilpotent subalgebra $\mathfrak{H}$ such that $\mathfrak{H}$ is the nilspace of (the $\mathfrak{G}$ part of) $\operatorname{ad} \mathfrak{H}$. Any representation of $\mathfrak{G}$ which vanishes on a Cartan subalgebra vanishes on $\mathfrak{G}$. Any maximal abelian ad reductive subalgebra of a semi-simple Lie algebra $\mathfrak{G}$ is a Cartan subalgebra and vice versa. If $\mathfrak{H}$ is a Cartan subalgebra of a semi-simple Lie algebra $\mathfrak{G}$ and ρ is a representation of $\mathfrak{G}$, then $\rho(\mathfrak{H})$ is fully reducible.

If $\mathfrak{G}$ is a Lie algebra with radical $\mathfrak{R}$, then the radical of the commutator subalgebra is easily seen from Levi's decomposition to be in the ideal $[\mathfrak{G}, \mathfrak{R}]$. It follows readily from Lie's theorem on the simultaneous triangularizing of matrices with coefficients in an algebraically closed field that the commutator subalgebra of $\mathrm{ad}(Kg + \mathfrak{R})$ consists of nilpotent elements for $g \varepsilon \mathfrak{G}$. It follows at once that $[\mathfrak{G}, \mathfrak{R}]$ is a nilpotent ideal of $\mathfrak{G}$ and thus the radical of the derived algebra of a Lie algebra is nilpotent. If $\mathfrak{N}$ is a subalgebra of a Lie algebra $\mathfrak{G}$, we mean by an *inner automorphism from* $\mathfrak{N}$ any product of automorphisms of the form $\exp(\mathrm{ad}\, x)$ with x in $\mathfrak{N}$ and $\mathrm{ad}\, x$ nilpotent (there is no difficulty concerning convergence and it can be verified readily that $\exp \mathrm{ad}\, x$ is an automorphism of $\mathfrak{G}$). The inner automorphism $\exp(\mathrm{ad}\, x)$ is a similarity in view of the identity $(\exp(\mathrm{ad}\, x))(y) = (\exp x) \cdot y \cdot (\exp x)^{-1}$.

The Cartan subalgebras of a *solvable* Lie algebra $\mathfrak{G}$ are conjugate under inner automorphisms from $\mathfrak{G}^{\infty}$ (cf. [1], [8]).

2. 3. Groups of unipotent elements. By the replica of an endomorphism x of a linear space is meant an element in the smallest algebraic Lie algebra containing x. A linear Lie algebra is algebraic if and only if it contains all the replicas of its elements (cf. [2]). The replicas of a nilpotent endomorphism n are precisely the multiples of n ([2], p. 159). Thus any linear Lie algebra of nilpotent endomorphisms is algebraic. If n is a nilpotent endomorphism, the connected algebraic Lie group determined by Kn is $\exp Kn$ ([2], p. 159). If u is a unipotent automorphism, its algebraic group hull is $\exp K \log u$ (cf. [2], p. 183) which is connected. Thus every unipotent element of an algebraic group lies in the connected component of the identity.

We shall mean by a rational representation of an algebraic Lie algebra $\mathfrak{G}$ a homomorphism which is the differential at the identity of a rational representation of an algebraic group G whose Lie algebra is $\mathfrak{G}$. Clearly if ρ is a rational representation of $\mathfrak{G}$ into $\mathfrak{H}$, and $\mathfrak{H}_1$ is an algebraic subalgebra of $\mathfrak{H}$, then $\rho^{-1}(\mathfrak{H}_1)$ is algebraic. Conversely, if $\mathfrak{G}$ is algebraic and ρ is rational, then $\rho(\mathfrak{G})$ is algebraic (though this is not always true when the characteristic of the base field is not 0) (cf. [2], pp. 140, 146).

The analogue of these results for algebraic groups is not true; that is, if G is an algebraic group and ρ a rational representation, then in general $\rho(G)$ is not algebraic. However in case G is an algebraic group of unipotent elements, the analogue is true in view of the following special features of such groups.[2]

[2] U1, U2, and U3 were proved by the author in the original version of this paper.

U1) An algebraic group of unipotent elements is connected and its Lie algebra consists of nilpotent elements.

U2) Let $\mathfrak{N}$ be a linear Lie algebra of nilpotent elements and let $X_1, \cdots, X_r$ be a base for $\mathfrak{N}$ such that the linear span $\mathfrak{N}_i$ of $X_i, \cdots, X_r$ is an ideal in $\mathfrak{N}_{i-1}$ $(i = 2, \cdots, r)$. Such a base always exists. Let N be a connected algebraic group with Lie algebra $\mathfrak{N}$. Then the elements of N can be expressed uniquely as $\exp(t_1X_1) \cdot \exp(t_2X_2) \cdots \exp(t_rX_r)$ with $t_1, \cdots, t_r$ in the ground field. ([3], Ch. V, Sec. 3, Prop. 17).

U3) Let ρ be a rational representation of an algebraic group N whose Lie algebra consists of nilpotent elements. Then $\rho(N)$ is algebraic ([3], Ch. V, Sec. 3, Prop. 15).

It appears from these results that an algebraic group consists of unipotent elements if and only if it is connected and its Lie algebra consists of nilpotent elements; in addition the image of such a group under a rational representation is algebraic.

Section 3. Rational Representations.

This section is devoted to a discussion of the image of fully reducible groups and groups of unipotent elements under a rational representation.

It is easily seen that a normal subgroup of a fully reducible linear group is fully reducible. We now prove a simple but highly useful partial converse.

LEMMA 3.1.[3] *Let N be a normal subgroup of finite index in the linear group G. G is fully reducible if and only if N is fully reducible.*

Proof. In view of the preceding remark, it remains only to prove that if N is fully reducible, then G is fully reducible.

Suppose therefore that N is fully reducible and that the finite set of elements $g_1, \cdots, g_p$ is a complete set of representatives for the cosets of $G \bmod N$. Let G operate on the linear space V and suppose it keeps invariant the subspace W. Let U be a complement to W invariant under N. For

After the paper was submitted for publication, the author learned that these results as well as many other interemediate results of the paper had been obtained independently by C. Chevalley and included in his "Theorie des Groupes de Lie," vol. 3, which was to appear soon. In the present revised version, the author has omitted most proofs of such results, taking advantage of Chevalley's Volume 3 as a reference.

[3] This observation seems to have gone unnoticed in the literature. It was noted independently by C. Chevalley who gives a different proof for it in Prop. 1, Ch. IV, Sec. 5 of [3].

each u in U, let u_i be the unique element of g_iU such that $u_i - u$ is in W $(i = 1, \cdots, p)$. Since N is normal in G, each g_iU is invariant under N and the finite set of subspaces $g_1U, \cdots, g_pU$ is permuted by the operations of G. As a result the finite (unordered) sets of elements $S_u = (u_1, \cdots, u_p)$ are permuted among themselves by the operations of G (according to the rule $gS_u = S_{u_g}$ where u_g is the element of U congruent to $gu \bmod W$). For each u in U, let $\bar{u} = (u_1 + \cdots + u_p)/p$ and let $\bar{U}$ denote the set of all elements $\bar{u}$. Then clearly $\bar{U}$ is a linear subspace complementary to W and invariant under G.

Combining Lemma 3.1 with the fact that a connected algebraic group is fully reducible if and only if its Lie algebra is, we conclude

PROPOSITION 3.1.[4] An algebraic group is fully reducible if and only if its Lie algebra is fully reducible.

PROPOSITION 3.2. *A rational representation of an algebraic group sends unipotent elements into unipotent elements and fully reducible subgroups into fully reducible subgroups.*[5]

Proof. It becomes clear after extending the ground field to its algebraic closure that no generality is lost if we assume the ground field to be algebraically closed. Accordingly we assume the ground field K to be algebraically closed. Let u be a unipotent element of the group and let $n = \log u$, which is in the Lie algebra of the group (cf. Sec. 2.2). Let ρ be the given rational representation and let $d\rho$ be the differential of ρ at the identity. We show that $d\rho(n)$ is nilpotent. We let P denote the field of formal power series in an indeterminate t with coefficients in K, and we extend the ground field to P. Let V_1 be the underlying linear space over K on which n acts, let E_1 denote the space of endomorphisms of V_1, and let V_1^P, E_1^P denote the extensions obtained on extending K to P. Let V_2, E_2, V_2^P, E_2^P be the spaces related similarly to the endomorphism $d\rho(n)$. For any element x of E_i^P we denote by $L(x)$ the set of values taken at x by the linear extensions to E_i^P of linear functions on E_i $(i = 1, 2)$. $L(x)$ can also be described as the K-linear span of the matrix coefficients of x relative to a base B, where the elements of B are in V_i $(i = 1, 2)$. We consider (cf. [2], Sec. 2, Theorem 9, p. 157) the relation

$$\text{(A)} \qquad \rho(\exp tn) = \exp t(d\rho(n)).$$

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[4] Also contained in [3].

[5] The assertion about unipotent elements follows easily from Prop. 16, p. 126 of [3].

Since n is nilpotent, $L(\exp tn)$ consists of polynomials in t with coefficients in K. Since ρ is rational map, $L(\rho(\exp tn))$ consists of rational functions of t. On the other hand, let b be an eigenvalue of $d\rho(n)$. Selecting a base in V_2 so as to make the matrix of $d\rho(n)$ triangular, we see that $L(\exp t(d\rho(n)))$ contains the power series $\exp bt$. In view of the cited identity, $\exp bt$ is a rational function in t, which is absurd if $b \neq 0$. (For if $\exp bt = f/g$ with f and g polynomials in t and $b \neq 0$, we arrive at the contradictions degree $g^2 f = \text{degree}\, g \cdot dg/dt \cdot f$ upon differentiating both sides). Hence $b = 0$ and $d\rho(n)$ is nilpotent.

The proof that ρ sends fully reducible subgroups into fully reducible subgroups runs as follows. In view of Proposition 3.1, it suffices to prove that $d\rho$ carries fully reducible Lie algebras into fully reducible Lie algebras. In view of the fact that a fully reducible Lie algebra is the direct sum of a semi-simple Lie algebra and an abelian algebra of fully reducible elements (cf. [6]) and that moreover every semi-simple linear algebra is fully reducible, ([11]), it suffices to prove that $d\rho$ carries fully reducible elements into fully reducible elements. Suppose then that X is a fully reducible endomorphism of V_1. Let $s + n$ be a Jordan sum decomposition of $Y = d\rho(X)$. Let $a_1, \cdots, a_n$ be the eigenvalues of X and $b_1, \cdots, b_n$ the eigenvalues of s. Upon diagonalizing X it is seen that $L(\exp tX)$ is the linear span of $\exp a_1 t, \cdots, \exp a_n t$. Suppose now $n \neq 0$. Upon putting Y into Jordan normal form, it is seen that $L(\exp tY)$ contains a term $t(\exp b_i t)$. Equation (A) above implies that $t(\exp b_i t)$ is a rational combination of $\exp a_1 t, \cdots,$ $\exp a_n t$—which is absurd (cf. [2], p. 151). The proof is now complete. Our proof has also established

PROPOSITION 3.3. *A rational representation of an algebraic Lie algebra carries fully reducible subalgebras into fully reducible subalgebras and nilpotent elements into nilpotent elements.*

There is a very elegant proof due to M. Schiffer of the First Main Theorem of the Theory of Invariants for linear groups all of whose "powers" or tensor representations are fully reducible (cf. [9], p. 300). Inasmuch as tensor representations are rational, we have as a consequence of Proposition 3.2.

THEOREM. *The First Main Theorem of the Theory of Invariants is valid for fully reducible groups.*

Section 4. Conjugacy of Fully Reducible Subalgebras.

Throughout this section, $\mathfrak{G}$ denotes a linear Lie algebra, $\mathfrak{R}$ its radical, and $\mathfrak{N}$ the set of nilpotent endomorphisms in $\mathfrak{R}$.

4.1. If G is an algebraic Lie algebra, it is known that it contains a semi-simple Lie subalgebra $\mathfrak{L}$ and an abelian subalgebra $\mathfrak{A}$ of fully reducible endomorphisms such that $G = \mathfrak{L} + \mathfrak{A} + \mathfrak{N}$, $[\mathfrak{L}, \mathfrak{A}] = 0$, and $\mathfrak{R} = \mathfrak{A} + \mathfrak{N}$ (semi-direct) (cf. [4]). Inasmuch as $\mathfrak{L}$ is fully reducible ([11]), it is easily seen that $\mathfrak{L} + \mathfrak{A}$ is fully reducible. Now $\mathfrak{N}$ can clearly be characterized as the maximal ideal of nilpotent endomorphisms (by applying in succession Engel's theorem and Lie's theorem on triangularizing solvable linear Lie algebras). Consequently a criterion that an algebraic Lie algebra be fully reducible is that it contain non non-zero ideal of nilpotent elements. It may be noted that $\mathfrak{N}$ is the totality of nilpotent endomorphisms in the radical.

4.2. If A, B are subspaces of a linear space E with $A \subset B$, then the totality of endomorphisms of E which send B into A is an algebraic Lie algebra ([2]). Coupling this fact with the fact that the adjoint representation of an algebraic Lie algebra is rational, we see: If $\mathfrak{G} \supset \mathfrak{B} \supset \mathfrak{A}$ are linear Lie algebras with $[\mathfrak{G}, \mathfrak{B}] \subset \mathfrak{A}$, then $[\mathfrak{G}^*, \mathfrak{B}] \subset \mathfrak{A}$, where $\mathfrak{G}^*$ denotes the smallest algebraic Lie algebra containing $\mathfrak{G}$.

If $\mathfrak{G} \supset \mathfrak{A}$ are linear Lie algebras, the *idealizer* of $\mathfrak{A}$ in $\mathfrak{G}$ is defined as the totality of elements x in G with $[x, \mathfrak{A}] \subset \mathfrak{A}$. If $\mathfrak{G}$ is algebraic, then by the foregoing the idealizer of $\mathfrak{A}$ in $\mathfrak{G}$ is algebraic. By simply "the idealizer of $\mathfrak{A}$" is meant the idealizer of $\mathfrak{A}$ in the totality of endomorphisms.

The radical of an algebraic Lie algebra is algebraic and so also is any subalgebra of nilpotent endomorphisms (cf. [3]).

4.3. An ideal $\mathfrak{B}$ of a fully reducible linear Lie algebra $\mathfrak{G}$ is fully reducible. For letting $\mathfrak{B}^*$ and $\mathfrak{G}^*$ denote the smallest algebraic Lie algebras containing $\mathfrak{B}$ and $\mathfrak{G}$, it is seen in turn that

$$[\mathfrak{B}, \mathfrak{G}^*] \subset [\mathfrak{B}, \mathfrak{G}] \subset \mathfrak{B}, \qquad [\mathfrak{B}^*, \mathfrak{G}^*] \subset [\mathfrak{B}, \mathfrak{G}^*] \subset \mathfrak{B} \subset \mathfrak{B}^*,$$

and therefore $\mathfrak{B}^*$ is an ideal of $\mathfrak{G}^*$. It is also seen that $\mathfrak{G}^*$ is fully reducible and $\mathfrak{B}$ is fully reducible if and only if $\mathfrak{B}^*$ is. Now the maximum ideal of nilpotent elements of $\mathfrak{B}^*$ is an ideal in $\mathfrak{G}^*$ and hence zero. Therefore $\mathfrak{B}^*$ is fully reducible and thus $\mathfrak{B}$ is too.

Any abelian ad-reductive subalgebra of a Lie algebra can be extended to a Cartan subalgebra (cf. [1] or [3]).

4.4. Suppose $\mathfrak{M}$ and $\mathfrak{N}$ are subalgebras which span the linear Lie algebra $\mathfrak{G}$. Suppose in addition that $\mathfrak{N}$ is an ideal and algebraic. Then $\mathfrak{G}$ is algebraic if and only if $\mathfrak{M}$ is.

Proof. Let ρ be a rational representation of $\mathfrak{F}$, the idealizer of $\mathfrak{N}$, whose kernel is $\mathfrak{N}$. (That such a representation exists is stated in [2] and proved in [3]). Then $\mathfrak{G} = \rho^{-1}(\rho(\mathfrak{M}))$ is algebraic according as $\mathfrak{M}$ is.

Lemma 4.1. *Let ρ be a representation of a solvable Lie algebra $\mathfrak{G}$. Then any two maximal ρ-reductive subalgebras of $\mathfrak{G}$ are conjugate under an inner automorphism from $\mathfrak{G}^\infty$. If ρ is an isomorphism, a ρ-reductive subalgebra is abelian.*

Proof. Since $\rho(\mathfrak{G}^\infty) = \rho(\mathfrak{G})^\infty$, it suffices to prove the theorem for a solvable linear Lie algebra $\mathfrak{G}$ (with ρ the identity map). Let $\mathfrak{A}_i$ $(i = 1, 2)$ be a maximal fully reducible subalgebra of $\mathfrak{G}$. $[\mathfrak{A}_i, \mathfrak{A}_i]$ consists of elements which are fully reducible and nilpotent (by Lie's triangularizing Theorem) and is thus (0); that is $\mathfrak{A}_i$ is abelian. Since $\mathfrak{A}_i$ is an abelian set of ad-reductive elements, it is an ad-reductive subalgebra. By 4.3 it can be extended to a Cartan subalgebra $\mathfrak{H}_i$ $(i = 1, 2)$. $\mathfrak{G}$ being solvable, there is an inner automorphism x from $\mathfrak{G}^\infty$ which sends $\mathfrak{H}_1$ into $\mathfrak{H}_2$ (cf. [1]). Now $\mathfrak{H}_i$ being a nilpotent linear Lie algebra, the fully reducible elements in it are simultaneously ad-reductive and ad-nilpotent and hence central. Hence they form a subalgebra—in fact $\mathfrak{A}_i$. The automorphism x which carries $\mathfrak{H}_1$ into $\mathfrak{H}_2$ carries $\mathfrak{A}_1$ into $\mathfrak{A}_2$ since it sends fully reducible elements into fully reducible elements (for it is a similarity). Proof of the lemma is now complete.

Lemma 4.2. *Let $\mathfrak{G}$ be a linear Lie algebra, $\mathfrak{N}$ an ideal of nilpotent elements in $\mathfrak{G}$, and $\mathfrak{M}$, $\mathfrak{M}_1$ fully reducible subalgebras of $\mathfrak{G}$ with $\mathfrak{M}_1 \subset \mathfrak{M} + \mathfrak{N}$. Then $\mathfrak{M} \cap (\mathfrak{M}_1 + \mathfrak{N})$ is fully reducible.*

Proof. Let $\mathfrak{F}$ be the idealizer of $\mathfrak{N}$. It is algebraic and contains $\mathfrak{G}$. Let ρ be a rational representation of $\mathfrak{F}$ with kernel $\mathfrak{N}$, and let ρ_0 be the restriction of ρ to $\mathfrak{M}^*$, the smallest algebraic Lie algebra containing $\mathfrak{M}$. Since $\mathfrak{M}^*$ is fully reducible and $\mathfrak{M}^* \cap \mathfrak{N}$ is an ideal of nilpotent elements in $\mathfrak{M}^*$, it is zero (by Remark 4.1). Hence ρ is one-to-one on $\mathfrak{M}^*$. Furthermore, since rational representations carry algebraic Lie algebras into algebraic Lie algebras and since inverse images under rational representations of algebraic

Lie algebras are algebraic, we have $\rho_0(\mathfrak{M}^*_2) = \rho_0(\mathfrak{M}_2)^*$ where $\mathfrak{M}_2$ is the subalgebra $\mathfrak{M} \cap (\mathfrak{M}_1 + \mathfrak{N})$ and the asterisk denotes closure in the sense of smallest containing algebraic Lie algebra. Now it is clear that $\rho_0(\mathfrak{M}_2) = \rho(\mathfrak{M}_2) = \rho(\mathfrak{M}_1)$ and hence $\rho_0(\mathfrak{M}_2)$ is fully reducible (Proposition 3.3). Consequently, $\rho_0(\mathfrak{M}^*_2) = \rho_0(\mathfrak{M}_2)^*$ is fully reducible and hence $\rho_0(\mathfrak{M}_2)^*$ contains no non-zero ideal of nilpotent elements by Remark 4.1. Since ρ_0 takes nilpotent elements into nilpotent elements and it is one-to-one on $\mathfrak{M}^*_2$, we conclude that $\mathfrak{M}^*_2$ contains no ideal of nilpotent elements. It follows that $\mathfrak{M}^*_2$ is fully reducible and hence $\mathfrak{M}_2$ is fully reducible.

THEOREM 4.1. *Any two maximal fully reducible subalgebras of a linear Lie algebra $\mathfrak{G}$ are conjugate under an inner automorphism from the radical of* $[\mathfrak{G}, \mathfrak{G}]$.

Proof. Let $\mathfrak{G}^*$ be the smallest algebraic Lie algebra containing $\mathfrak{G}$ and let $\mathfrak{M}$ be a fully reducible subalgebra of $\mathfrak{G}^*$. Then $\mathfrak{M} \cap \mathfrak{G}$ is an ideal in $\mathfrak{M}$ and hence fully reducible (Remark 4.1). Again since $[\mathfrak{G}^*, \mathfrak{G}^*] = [\mathfrak{G}, \mathfrak{G}]$, and moreover some maximal fully reducible subalgebra of $\mathfrak{G}^*$ contains a maximal fully reducible subalgebra of $\mathfrak{G}$, Theorem 4.1 would be valid for $\mathfrak{G}$ if it were valid for algebraic Lie algebras.

Thus we may adopt the additional hypothesis that $\mathfrak{G}$ is algebraic without loss of generality. We proceed by induction on the dimension of $\mathfrak{G}$.

Let $\mathfrak{N}$ denote the ideal of nilpotent elements in the radical of the algebraic Lie algebra $\mathfrak{G}$, and let $\mathfrak{M}_1$ be a fully reducible subalgebra such that $\mathfrak{G} = \mathfrak{M}_1 + \mathfrak{N}$ (cf. 4.1). $\mathfrak{M}_1$ is a maximal fully reducible subalgebra, since any fully reducible algebra $\mathfrak{M}_0$ containing it must intersect $\mathfrak{N}$ in an ideal of $\mathfrak{M}_0$, and hence (Remark 4.1) in zero. Furthermore, the radical of $[\mathfrak{G}, \mathfrak{G}]$ is $[\mathfrak{M}_1, \mathfrak{N}]$ (Remark 4.1). Let now $\mathfrak{M}_2$ be a maximal fully reducible subalgebra of $\mathfrak{G}$. Then $\mathfrak{M}_1 \cap (\mathfrak{M}_2 + \mathfrak{N})$ is fully reducible by Lemma 4.2, indeed maximal fully reducible in $\mathfrak{M}_2 + \mathfrak{N}$ since it is complementary to $\mathfrak{N}$ in $\mathfrak{M}_2 + \mathfrak{N}$. Furthermore $\mathfrak{M}_2$ is algebraic and hence $\mathfrak{M}_2 + \mathfrak{N}$ is algebraic. Hence we must have $\mathfrak{M}_2 + \mathfrak{N} = \mathfrak{G}$; otherwise on applying the induction assumption we would be led to the false conclusion that $\mathfrak{M}_1 \cap (\mathfrak{M}_2 + \mathfrak{N})$ is conjugate to a maximal fully reducible subalgebra of $\mathfrak{G}$ and hence is $\mathfrak{M}_1$. Now let $\mathfrak{L}_i + \mathfrak{A}_i$ be a Levi-decomposition for the fully reducible subalgebra $\mathfrak{M}_i$ $(i = 1, 2)$, $\mathfrak{L}_i$ being semi-simple. Then it is easily seen that $\mathfrak{L}_i$ is a maximal semi-simple subalgebra of $\mathfrak{G}$. As is well-known, the maximal semi-simple subalgebras of a Lie algebra (over a field of characteristic zero) are conjugate under an inner automorphism from the radical of the commutator subalgebra (cf. [5]). Carrying $\mathfrak{L}_2$

into $\mathfrak{L}_1$ by such an automorphism, we see that no generality is lost in assuming $\mathfrak{L}_1 = \mathfrak{L}_2$. Again, since $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are each contained in the idealizer of $\mathfrak{L}_1$ in $\mathfrak{G}$, no generality is lost in assuming that $\mathfrak{L}_1$ is an ideal in $\mathfrak{G}$. In this case, $\mathfrak{G}$ is the direct sum of $\mathfrak{L}_1$ and its radical $\mathfrak{R}$. Now $\mathfrak{M}_i$ being maximal fully reducible in $\mathfrak{G}$, and fully reducible subalgebras of $\mathfrak{R}$ being abelian and commutative with $\mathfrak{L}_1$, it is easily seen that $\mathfrak{M}_i \cap \mathfrak{R}$ is a maximal fully reducible subalgebra of $\mathfrak{R}$. Applying Lemma 4.1 to $\mathfrak{R}$, we can get an inner automorphism from $\mathfrak{R} \cap [\mathfrak{G}, \mathfrak{G}]$ which sends $\mathfrak{M}_2$ into $\mathfrak{M}_1$. The proof of the theorem is now complete.

COROLLARY 4.1. *Let $\mathfrak{G}$ be a linear Lie algebra with radical $\mathfrak{R}$. Assume $\mathfrak{A}$ is a maximal fully reducible abelian subalgebra of $\mathfrak{G}$. Then there is a maximal semi-simple subalgebra $\mathfrak{L}$ of $\mathfrak{G}$ such that $\mathfrak{A} = \mathfrak{A} \cap \mathfrak{L} + \mathfrak{A} \cap \mathfrak{R}$ with (1) $\mathfrak{A} \cap \mathfrak{L}$ a Cartan subalgebra of $\mathfrak{L}$, (2) $\mathfrak{A} \cap \mathfrak{R}$ a maximal fully reducible subalgebra of $\mathfrak{R}$ and (3) $[\mathfrak{L}, \mathfrak{A} \cap \mathfrak{R}] = 0$.*

Proof. Let $\mathfrak{B}$ be a maximal fully reducible subalgebra of $\mathfrak{R}$. Since $\mathfrak{R}$ is an ideal in $\mathfrak{G}$, we have $\mathfrak{G} = Z(\mathfrak{B}) + \mathfrak{R}$ where $Z(\mathfrak{B})$ is the centralizer of $\mathfrak{B}$. Letting $\mathfrak{L}$ be the semi-simple component in a Levi-decomposition for $Z(\mathfrak{B})$, we see that $\mathfrak{G} = \mathfrak{L} + \mathfrak{R}$ and $\mathfrak{L}$ is a maximal semi-simple subalgebra of $\mathfrak{G}$. Moreover, $\mathfrak{L} + \mathfrak{B}$ is readily seen to be maximal fully reducible in $\mathfrak{G}$. Any maximal fully reducible abelian subalgebra of $\mathfrak{L} + \mathfrak{B}$ clearly has the properties (1), (2), (3) of the Corollary. Inasmuch as the maximal fully reducible subalgebras are conjugate under inner automorphisms, any maximal fully reducible abelian subalgebra has the desired properties.

Another consequence, which follows directly from the theorem is

COROLLARY 4.2. *Any two maximal ad-reductive subalgebras of a Lie algebra are conjugate under an inner automorphism from the radical of the commutator subalgebra.*

Section 5. Invariant Subalgebras of a Fully Reducible Group of Automorphisms.

LEMMA 5.1. *Let $\mathfrak{M}$ be a subalgebra and $\mathfrak{N}$ an abelian ideal of a Lie algebra $\mathfrak{G}$, and let Γ be a group of automorphisms of $\mathfrak{G}$. Assume*

1) $\mathfrak{G} = \mathfrak{M} + \mathfrak{N}$ *semi-directly;*

2) *for each automorphism T of Γ there is an element n in $\mathfrak{N}$ such that* $T(\mathfrak{M}) = \exp \operatorname{ad} n(\mathfrak{M})$;

3) Γ *keep* $\mathfrak{N}$ *invariant.*

Then Γ *keeps invariant some image of* $\mathfrak{M}$ *by an inner automorphism.*

Proof. Let A denote the totality of linear subspaces of $\mathfrak{G}$ that are complementary to $\mathfrak{N}$. We regard A as an affine space in the natural way; viz, if we regard $\mathfrak{M}$ as the "origin" or "reference point" of A, then the linear maps of $\mathfrak{M}$ into $\mathfrak{N}$ can be identified with the associated vector space V of A under the correspondence $\mathfrak{L} - \mathfrak{M} \rightarrow \phi(\mathfrak{L})$ where $\mathfrak{L} \varepsilon A$ and $\phi(\mathfrak{L})(m)$ is the element p in $\mathfrak{N}$ such that $m + p \varepsilon \mathfrak{L}$.

The totality S of images of $\mathfrak{M}$ by inner automorphisms forms an affine subspace of A. For $\mathfrak{N}$ being an abelian ideal, $(\operatorname{ad} n)^2 = 0$ for n in N and thus $\exp \operatorname{ad} n(m) = m + [n, m]$; consequently the family S is identified with the linear family of maps $m \rightarrow [n, m]$ (n varying over $\mathfrak{N}$) of $\mathfrak{M}$ into $\mathfrak{N}$.

Now the fully reducible group of automorphisms Γ of $\mathfrak{G}$ induces on A a group of affine transformations and has in A a fixed point $\mathfrak{M}_0$. Selecting $\mathfrak{M}_0$ as origin in A, we obtain a representation of the group of automorphisms of $\mathfrak{G}$ which keep $\mathfrak{M}_0$ invariant on the vector space V associated with A and this representation is easily seen to be rational. Since a rational representation takes fully reducible groups into fully reducible groups, it follows that Γ^* the image of Γ under this representation is fully reducible. Furthermore the operations of Γ on A keep invariant the affine subspace S. Hence Γ^* keeps invariant the linear subspace $U = S - \mathfrak{M}$ of V. Being fully reducible, Γ^* keeps invariant a subspace W complementary to U. Hence the operations of Γ on A keep invariant the affine subspace $\mathfrak{M}_0 + W$, which intersects $S = \mathfrak{M} + U$ in but a single point—call it $\mathfrak{M}_1$. Obviously $\mathfrak{M}_1$ is a fixed point of Γ^* and correspondingly the subalgebra $\mathfrak{M}_1$ is invariant under Γ and is the image of $\mathfrak{M}$ under an inner automorphism from $\mathfrak{N}$. Proof of the lemma is now complete.

Note. The analogue of Lemma 5.1 for associative algebras is true and proof is exactly the same.

Definition. *Let* $\mathfrak{G}$ *be a linear Lie algebra, let* $\mathfrak{N}$ *be an ideal of* $\mathfrak{G}$, *and let* T *be a similarity of* $\mathfrak{G}$ *which keeps* $\mathfrak{N}$ *invariant. Let* ρ *be a rational representation of* $\mathfrak{G}$ *with kernel* $\mathfrak{N}$, *and let* T_1 *denote the automorphism of* $\rho(G)$ *induced by* T. *Then* T_1 *is called an s-automorphism of* $\rho(G)$.

Clearly an s-automorphism of a linear Lie algebra carries maximal fully reducible subalgebras into maximal fully reducible subalgebras and nilpotent elements into nilpotent elements.

THEOREM 5.1. *A pre-fully reducible group of similarities of a linear Lie algebra keeps invariant a maximal fully reducible subalgebra.*

Proof. Let $\mathfrak{G}$ be the linear Lie algebra and $\mathfrak{G}^*$ the smallest algebraic Lie algebra containing $\mathfrak{G}$.

We remark first that any maximal fully reducible subalgebra of $\mathfrak{G}^*$ intersects $\mathfrak{G}$ in a maximal fully reducible subalgebra of $\mathfrak{G}$. For obviously some maximal fully reducible subalgebra $\mathfrak{M}^*$ of $\mathfrak{G}^*$ includes a maximal fully reducible subalgebra $\mathfrak{M}$ of $\mathfrak{G}$. Since $[\mathfrak{G}^*, \mathfrak{G}^*] = [\mathfrak{G}, \mathfrak{G}]$, $\mathfrak{M}^* \cap \mathfrak{G}$ is an ideal in $\mathfrak{M}^*$ and fully reducible by Remark 4.3. Hence $\mathfrak{M}^* \cap \mathfrak{G} = \mathfrak{M}$. Inasmuch as any maximal fully reducible subalgebra of $\mathfrak{G}^*$ is conjugate to $\mathfrak{M}^*$ by an inner automorphism from $[\mathfrak{G}^*, \mathfrak{G}^*] \subset \mathfrak{G}$, it follows at once that *every* maximal fully reducible subalgebra of $\mathfrak{G}^*$ intersects $\mathfrak{G}$ in a maximal fully reducible subalgebra of $\mathfrak{G}$.

Next we note that any similarity T keeping $\mathfrak{G}$ invariant also keeps $\mathfrak{G}^*$ invariant. For $T(\mathfrak{G}^*) \cap \mathfrak{G}^*$ is algebraic and includes $\mathfrak{G}$. Hence it includes $\mathfrak{G}^*$; consequently $T(\mathfrak{G}^*) = \mathfrak{G}^*$.

In view of the preceding observation, we can assume without loss of generality that $\mathfrak{G}$ is an algebraic Lie algebra. Thus the theorem will be established when we will have proved: a fully reducible group of s-automorphisms of an algebraic Lie algebra $\mathfrak{G}$ keeps invariant a maximal fully reducible subalgebra of $\mathfrak{G}$.

We decompose $\mathfrak{G}$ into $\mathfrak{M} + \mathfrak{N}$ where $\mathfrak{M}$ is a maximal fully reducible subalgebra and $\mathfrak{N}$ is the maximum ideal of nilpotent elements. We prove the above assertion by induction on $r = \dim[\mathfrak{N}, \mathfrak{N}]$.

If $\dim[\mathfrak{N}, \mathfrak{N}] = 0$, the assertion follows immediately from Lemma 5.1. Assume now that $\dim[\mathfrak{N}, \mathfrak{N}] > 0$ and that the induction hypothesis holds for $r < \dim[\mathfrak{N}, \mathfrak{N}]$. $[\mathfrak{N}, \mathfrak{N}]$ is an algebraic Lie algebra. Hence there is a rational representation ρ of $\mathfrak{G}$ whose kernel is $[\mathfrak{N}, \mathfrak{N}]$. Applying the induction hypothesis to the induced fully reducible group of s-automorphisms of $\rho(\mathfrak{G})$, we are reduced to verifying the hypothesis for $\mathfrak{M}_1 + [\mathfrak{N}, \mathfrak{N}]$ where $\mathfrak{M}_1$ is the image of $\mathfrak{M}$ by an inner automorphism from $\mathfrak{N}$. Since an inner automorphism is a similarity, $\mathfrak{M}_1$ is a maximal fully reducible subalgebra of $\mathfrak{G}$ and hence algebraic. Consequently $\mathfrak{M}_1 + [\mathfrak{N}, \mathfrak{N}]$ is algebraic by 4.4. Furthermore $\mathfrak{N}$ is solvable by Engel's theorem so that the dimension of the commutator subalgebra of $[\mathfrak{N}, \mathfrak{N}]$ is smaller than $\dim[\mathfrak{N}, \mathfrak{N}]$. We can thus apply the induction hypothesis to $\mathfrak{M}_1 + [\mathfrak{N}, \mathfrak{N}]$ and thereby complete the proof of Theorem 5.1.

COROLLARY 5.1. *A fully reducible group of automorphisms of a Lie algebra keeps invariant a maximal ad-reductive subalgebra.*

Proof. Once we note that the center of a Lie algebra belongs to every maximal ad-reductive subalgebra, this corollary follows directly from Theorem 5.1.

COROLLARY 5.2. *A fully reducible group of automorphisms of a Lie algebra keeps invariant a maximal semi-simple subalgebra.*

Proof. Follows directly from Corollary 5.1 and the observation that an ad-reductive subalgebra has a unique maximum semi-simple subalgebra—viz., its commutator subalgebra.

THEOREM 5.2. *A fully reducible group of automorphisms of a solvable Lie algebra keeps invariant a Cartan subalgebra.*

Proof. Let $\mathfrak{G}$ be the solvable Lie algebra and Γ the fully reducible group of automorphisms. The operation $\mathrm{ad}\, X \to \mathrm{ad}\, c(X)$, $c \in \Gamma$, of c on $\mathrm{ad}\,\mathfrak{G}$ is a similarity since $\mathrm{ad}\, c(X) = c(\mathrm{ad}\, X)c^{-1}$. Inasmuch as any Cartan subalgebra of $\mathfrak{G}$ is the inverse image of a Cartan subalgebra of $\mathrm{ad}\,\mathfrak{G}$ (and conversely), it suffices to prove

(A) any pre-fully reducible group of similarities of a solvable linear Lie algebra keeps invariant a Cartan subalgebra.

Now it is not difficult to see that any Cartan subalgebra of a linear Lie algebra $\mathfrak{L}$ is the intersection with $\mathfrak{L}$ of a Cartan subalgebra of the algebraic hull $\mathfrak{L}^*$ and conversely (cf. [3]). Consequently it suffices to prove (A) under the additional hypothesis that the solvable linear Lie algebra is algebraic.

We can suppose accordingly that Γ is a pre-fully reducible group of similarities of a solvable algebraic Lie algebra $\mathfrak{G}$. By Corollary 5.1 Γ keeps invariant a maximal fully reducible subalgebra $\mathfrak{M}$ of $\mathfrak{G}$. Now any algebraic Lie algebra obviously contains some regular fully reducible element x and the nilspace of $(\mathrm{ad}\, x)_{\mathfrak{G}}$ is a Cartan subalgebra. Furthermore any fully reducible subalgebra $\mathfrak{F}$ of a solvable linear Lie algebra is abelian—for its ideal $[\mathfrak{F}, \mathfrak{F}]$ being fully reducible and consisting of nilpotent elements is zero. Thus any maximal ad-reductive subalgebra $\mathfrak{M}_x$ of $\mathfrak{G}$ which contains x extends to a unique Cartan subalgebra—the nilspace of $(\mathrm{ad}\,\mathfrak{M}_x)_{\mathfrak{G}}$ or the centralizer of $\mathfrak{M}_x$. Since all maximal fully reducible subalgebras of $\mathfrak{G}$ are conjugate, the centralizer of any maximal fully reducible subalgebra of $\mathfrak{G}$ is a Cartan

subalgebra. In particular, the centralizer of $\mathfrak{M}$ is a Cartan subalgebra invariant under Γ. Proof of the theorem is now complete.

By paralleling the proof of Theorem 5.1 we can prove

THEOREM. *A fully reducible group of automorphisms of an associative or Jordan algebra keeps invariant a maximal fully reducible subalgebra.*

The pertinent facts about Jordan algebras required for the above proof can be found in the paper by N. Jacobson in *Transactions of the American Mathematical Society*, vol. 70, (1951), p. 528.

Section 6. A Decomposition of Algebraic Groups.

Let G be an algebraic group. We denote by $\mathfrak{M}$ a maximal fully reducible subalgebra of its Lie algebra $\mathfrak{G}$, and by $\mathfrak{N}$ the ideal of nilpotent elements in its radical. We denote by M^A the totality of elements x in G such that $\operatorname{Ad} x$ keeps $\mathfrak{M}$ invariant, that is $x\mathfrak{M}x^{-1} = \mathfrak{M}$. Because of its maximal character, $\mathfrak{M}$ is algebraic, and because $\mathfrak{N}$ consists of nilpotent elements only, it, too, is algebraic ([2], pp. 181 and 183). We denote by N the connected group with Lie algebra $\mathfrak{N}$. N is normal in G since $\mathfrak{N}$ is invariant under all similarities from G.

LEMMA 6.1. *Let G be an algebraic group, M^A and N the subgroups of G defined above. Then $G = M^A \cdot N$.*

Proof. Since an algebraic group contains the Jordan product components of each of its elements, it suffices to prove that $M^A \cdot N$ contains each fully reducible and each unipotent element of G.

Suppose that u is unipotent. Then $u = \exp X$ with $\log u = X$ a nilpotent element in the Lie algebra $\mathfrak{G}$. Inasmuch as $KX + \mathfrak{N}$ is a solvable Lie algebra, by Lie's theorem on simultaneous triangularizing it consists of nilpotent elements. Let X_1 be a non-zero element of $(KX + \mathfrak{N}) \cap \mathfrak{M}$ if such exists, otherwise zero. Choosing a suitable base for $\mathfrak{N}$, we can apply U2 of 2.3 to conclude that $u = \exp(t_1X_1) \cdot u'$ with u' in N. Now $\exp t_1X_1$ belongs to the connected algebraic group with Lie algebra KX_1 and hence belongs to the connected algebraic group corresponding to $\mathfrak{M}$; hence $\operatorname{Ad}(\exp t_1X_1)(\mathfrak{M}) = \mathfrak{M}$ and $\exp t_1X_1$ is in M^A. It follows that u is in $M^A \cdot N$.

Suppose on the other hand that s is a fully reducible element of G. By Theorems 5.1 and 4.1 there exists an element u in N such that $s(uMu^{-1})s^{-1} = u\mathfrak{M}u^{-1}$. Hence $(u^{-1}su)\mathfrak{M}(u^{-1}su)^{-1} = \mathfrak{M}$ and $u^{-1}su = m$ is in

M^A. But $\mathfrak{N}$ being a normal subgroup of G, sus^{-1} is in N and hence $v = u^{-1}(sus^{-1})$ is in N. Since $vs = u^{-1}su = m$, we have $s = mv^{-1}$ is in $M^A \cdot N$. Proof of Lemma 6.1 is now complete.

THEOREM 6.1. *Let G be an algebraic group, $\mathfrak{M}$ a maximal fully reducible subalgebra of its Lie algebra $\mathfrak{G}$, and $\mathfrak{N}$ the ideal of nilpotent elements in the radical of $\mathfrak{G}$. There is a fully reducible algebraic group M with Lie algebra $\mathfrak{M}$ such that $G = M \cdot N$ (semi-direct), N being the connected algebraic group with Lie algebra $\mathfrak{N}$.*

Proof. By Lemma 6.1, $G = M^A \cdot N$ (not necessarily semi-direct). It is easily seen from the definition of M^A that it is an algebraic group whose Lie algebra $\mathfrak{G}_1$ contains $\mathfrak{M}$.

We first note that it is sufficient to prove Theorem 6.1 for the group M^A. For clearly $\mathfrak{G}_1 = \mathfrak{M} + (G_1 \cap \mathfrak{N})$ and $\mathfrak{G}_1 \cap \mathfrak{N}$ is the ideal of nilpotent element in the radical of $\mathfrak{G}_1$. If now $M^A = M \cdot N_1$ (semi-direct) where M and N_1 are algebraic groups with Lie algebras $\mathfrak{M}$ and $\mathfrak{G}_1 \cap \mathfrak{N}$ respectively, then $G = MN_1N = MN$. Moreover $M \cap N$, being an algebraic group with Lie algebra $\mathfrak{M} \cap \mathfrak{N} = (0)$, is a finite subgroup of N; being fully reducible it reduces to the identity element. Thus $G = M \cdot N$ (semi-direct).

Hence we lose no generality in assuming $G = M^A$. In this case $\mathfrak{M}$ is an ideal of $\mathfrak{G}$, and $\mathfrak{M}$ being ad-reductive (cf. Prop. 3.2) $\mathfrak{G} = \mathfrak{M} + \mathfrak{N}$ (direct). We proceed by induction on the dimension of $\mathfrak{G}$.

We let $\mathfrak{N}_1$ denote the center of $\mathfrak{N}$, and we denote by N_1 and M_0 the connected algebraic groups corresponding to $\mathfrak{N}_1$ and $\mathfrak{M}$ respectively. Clearly $\mathfrak{N}_1$ and $\mathfrak{M}$ are invariant under similarities from G, and hence N_1 and M_0 are normal subgroups of G. Furthermore, if $\mathfrak{N} \neq 0$ and $\mathfrak{G} \neq \mathfrak{N}_1$, there is a normal algebraic subgroup F of positive dimension such that the dimension of the subalgebra $\mathfrak{F} + \mathfrak{M}$ is smaller than $\dim \mathfrak{G}$ ($\mathfrak{F}$ being the Lie algebra of F)—namely $F = M_0$ or, if M_0 is zero dimensional $F = N_1$.

Case 1. $\mathfrak{N} = 0$. Here $\mathfrak{G} = \mathfrak{M}$ is fully reducible and hence G is fully reducible (Prop. 3.1).

Case 2. $\mathfrak{N} \neq 0$, $\mathfrak{G}$ is not the center of N. Select the subgroup F described above. Selecting a rational representation ρ with kernel F we know by Prop. 3.2 that $\mathfrak{M}' = d\rho(\mathfrak{M})$ is fully reducible, and $\mathfrak{N}' = d\rho(\mathfrak{N})$ is composed of nilpotent elements. By the induction hypothesis $G' = M' \cdot N'$ (semi-direct) where G' is the algebraic group hull of $\rho(G)$, M' is an algebraic subgroup with Lie algebra $\mathfrak{M}'$ and N' is the connected algebraic group with

Lie algebra $\mathfrak{N}'$. By U2 of 2.3 $\rho(N) = N'$. Hence $G = \rho^{-1}(M') \cdot N$. Applying the induction assumption to $\rho^{-1}(M')$, we obtain the desired fully reducible subgroup with Lie algebra $\mathfrak{M}$.

Case 3. $\mathfrak{G}$ is the center of $\mathfrak{N}$. Here $\mathfrak{G} = \mathfrak{N}$ is abelian, and G is a finite extension of N. Furthermore by U2 of 2.3 and the formula $\exp X \exp Y = \exp(X + Y)$ if $[X, Y] = 0$, we see that $X \rightarrow \exp X$ is an isomorphism of the additive group of the linear space $\mathfrak{N}$ with the multiplicative group N. Our theorem will therefore follow from the following: Let N be a vector group over a field K of characteristic zero (i.e. the additive group of a linear space over K—possibly infinite dimensional); then any finite extension of N splits. This in turn is the group extension interpretation of the now well known result: the two dimensional cohomology group of a finite group in an indefinitely divisible abelian group vanishes. (Indeed the cohomology groups vanish in all positive dimensions.)

Section 7. Conjugacy of Fully Reducible Subgroups.

THEOREM 7.1. *Let G be an algebraic group, $\mathfrak{N}$ the set of nilpotent elements in the radical of its Lie algebra, and N the connected algebraic group with Lie algebra $\mathfrak{N}$. Then $G = M \cdot N$ (semi-direct) with M a maximal fully reducible subgroup. Furthermore any two maximal fully reducible subgroups of G are conjugate under an inner automorphism from $\mathfrak{N}$.*

Proof. If M is a fully reducible subgroup of the algebraic group G with $G = M \cdot N$, then M is algebraic and *maximal* fully reducible and $G = M \cdot N$ (semi-direct). For let $\bar{M}$ be a maximal fully reducible subgroup containing M. $\bar{M}$ is clearly algebraic and its Lie algebra $\mathfrak{M}$ is fully reducible. Since $\mathfrak{M} \cap \mathfrak{N}$ is an ideal of nilpotent element in the radical of $\mathfrak{M}$, by Remark 4.1 it follows that $\mathfrak{M} \cap \mathfrak{N} = 0$. Since $\bar{M} \cap N$ is therefore a finite algebraic group, it consists of fully reducible unipotent elements and hence reduces to the identity. Thus $G = \bar{M} \cdot N$ (semi-direct) and $M = \bar{M}$.

We now prove the second part of the theorem by induction on the dimension of $\mathfrak{G}$, the Lie algebra of G. Let M_1 be a maximal fully reducible subgroup of G and $\mathfrak{M}_1$ its Lie algebra. Then $\mathfrak{F} = \mathfrak{M}_1 + \mathfrak{N}$ is algebraic (Sec. 4, Remark 4.4). Let F_0 denote the corresponding connected algebraic group and let F be the finite (algebraic) extension $M_1 F_0$. Now $G = M \cdot N$ implies that $F = (F \cap M) \cdot N$. Since $\mathfrak{F} \cap \mathfrak{M} = \mathfrak{M} \cap (\mathfrak{M}_1 + \mathfrak{N})$, the Lie algebra of $F \cap M$ is fully reducible by Lemma 4.4. Hence $F \cap M$ is fully reducible. By the opening remark of our proof, $F \cap M$ is a maximal fully reducible sub-

algebra of the algebraic group F. If $\dim \mathfrak{F}$ were less than $\mathfrak{G}$, we could apply the induction hypothesis to F and carry M_1 into $F \cap M$ by an inner automorphism from N and we would be led to the contradiction that $F \cap M$ is a maximal fully reducible subgroup of G. Consequently $\mathfrak{F} = \mathfrak{G}$, that is $\mathfrak{M}_1 + \mathfrak{N} = \mathfrak{M} + \mathfrak{N}$. From this it follows readily that the fully reducible subalgebra $\mathfrak{M}_1$ is maximal fully reducible in $\mathfrak{G}$.

By Theorem 4.1, $\mathfrak{M}_1$ and $\mathfrak{M}$ are conjugate under an inner automorphism from $\mathfrak{N}$. Hence no generality is lost in asuming $\mathfrak{M} = \mathfrak{M}_1$. Under this additional hypothesis, M_1 and M are each contained in M^A, the normalizer of $\mathfrak{M}$ in G. Furthermore the set of nilpotent elements in the radical of M^A is in $\mathfrak{N}$. Hence no generality is lost in assuming $G = M^A$. Repeating the 3-case argument in the proof of Theorem 6.1, we reduce to the case that $\mathfrak{G} = \mathfrak{N}$ is an abelian Lie algebra of nilpotent endomorphisms. If M_1 is a maximal fully reducible subgroup of G, $\mathfrak{M}_1$ consists of, in this case, fully reducible nilpotent elements and hence is 0. Thus M_1 is a finite group.[6] To prove that $u^{-1}M_1u \subset M$ for some u in N is equivalent to showing $M_1u \subset uM$; that is the group M_1 operating on the space of cosets G/M by left translation admits some fixed point uM. Since $G = M \cdot N$ (semi-direct), G/M can be identified with the space N. Under this identification, the operation of G on G/M by left translation is equivalent to the operation of G on the linear space $\mathfrak{N}$ by affine transformations in view of the identities

$$mnM = (mnm^{-1}) \cdot M \qquad \text{for } m \text{ in } M \tag{1}$$

$$n_1(nM) = (n_1 n) \cdot M \qquad \text{for } n_1 \text{ in } N \tag{2}$$

$$m \exp X m^{-1} = \exp mXm^{-1} \qquad \text{for } X \text{ in } N \tag{3}$$

$$\exp(X + Y) = \exp X \cdot \exp Y \qquad \text{for } X, Y \text{ in } \mathfrak{N} \tag{4}$$

together with the fact that $X \to \exp X$ is a one-to-one mapping of $\mathfrak{N}$ onto N. Now as is well-known, any finite group of affine transformations of a linear space (over a field of characteristic not dividing the order of the group) has a fixed point—namely, the centroid of any orbit. Hence M_1 has a fixed point on G/M and M_1 is conjugate to M under an inner automorphism from N. Proof of the theorem is now complete.

Corollary. *A pre-fully reducible group of similarities of an algebraic group keeps invariant a maximal fully reducible subgroup.*

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[6] At this point the theorem follows from the fact that the one dimensional cohomology group of a finite group with coefficients in an indefinitely divisible abelian group is zero. Indeed this is in essence the concluding argument of our proof.

Proof. Let G be the algebraic group. The pre-fully reducible group of similarities of G are the automorphisms $g \to fgf^{-1}$ where f varies over a fully reducible group F. Let G_1 be the normalizer of G, let M be a maximal fully reducible subgroup of G and let M_1 be a maximal fully reducible subgroup of G_1 which includes M. Since $F \subset G_1$, there is an element x in G_1 with $F \subset xM_1x^{-1}$ by Theorem 7.1. Hence for any $f \varepsilon F$ and $p \varepsilon xMx^{-1}$, $fpf^{-1} \varepsilon (xM_1x^{-1}) \cap G$. But xM_1x^{-1} being a fully reducible subgroup of G_1, its intersection with G is a fully reducible subgroup of G (cf. 4.3) and therefore coincides with xMx^{-1}. Thus F keeps invariant the maximal fully reducible subgroup xMx^{-1}.

Section 8. Relation to Wedderburn Decomposition.

THEOREM 8.1. *Let G be an algebraic group and let $\mathfrak{E}$ denote its enveloping associative algebra. Then there is a Wedderburn decomposition $\mathfrak{S} + \mathfrak{T}$ for $\mathfrak{E}$ with $\mathfrak{S}$ semi-simple and $\mathfrak{T}$ the radical such that:*

a) *$\mathfrak{S} \cap G$ is a maximal fully reducible subgroup of G,*

b) *$\mathfrak{T} \cap \mathfrak{G}$ is the ideal of nilpotent elements in the radical of $\mathfrak{G}$, the Lie algebra of G,*

c) *$(I + \mathfrak{T}) \cap G$ is the subgroup of unipotent elements in the radical of G (I denotes the identity element).*

Proof. Let $G = M \cdot N$ be a decomposition of G into the semi-direct product of a maximal fully reducible subgroup M and the subgroup N of the unipotent elements in the radical.

For any set of endomorphisms S we denote by $\mathfrak{E}(S)$ the enveloping associative algebra—i. e. polynomial combinations without constant term in the elements of S. We denote by $\mathfrak{L}(S)$ the linear span of S. If S is a group then $\mathfrak{E}(S)$ consists only of linear combinations of elements of S. Hence $M \cdot N = N \cdot M$ implies that

$$\mathfrak{L}(\mathfrak{E}(M) \cdot \mathfrak{E}(N)) = \mathfrak{L}(\mathfrak{E}(N) \cdot \mathfrak{E}(M)) = \mathfrak{E}(M \cdot N) = \mathfrak{E}(G).$$

We denote by $\log u$ for u a unipotent endomorphism the finite sum

$$\sum_{i=1}^{\infty} (-1)^{i+1} (u-1)^i / i.$$

Since $u = \exp \log u$, we have

$$\mathfrak{E}(N) = \mathfrak{E}(\log N + I) = \mathfrak{E}(\log N) + KI = \mathfrak{E}(\mathfrak{N}) + KI,$$

where $\mathfrak{N} = \log N$ is the Lie algebra of N and K is the ground field. As a result

$$\mathfrak{E}(G) = \mathfrak{L}(\mathfrak{E}(M) \cdot (\mathfrak{E}(\mathfrak{N}) + KI)) = \mathfrak{E}(M) + \mathfrak{L}(\mathfrak{E}(M) \cdot \mathfrak{E}(N)).$$

Now $\mathfrak{N}$ being an ideal in $\mathfrak{G}$, we have $mnm^{-1} \varepsilon \mathfrak{N}$ whenever $m \varepsilon M$ and $n \varepsilon \mathfrak{N}$ and as a result

$$M \cdot \mathfrak{N} = \mathfrak{N} \cdot M,\ \mathfrak{L}(\mathfrak{E}(M) \cdot \mathfrak{E}(\mathfrak{N})) = \mathfrak{L}(\mathfrak{E}(\mathfrak{N}) \cdot \mathfrak{E}(M)),$$

and from this it follows that $\mathfrak{L}(\mathfrak{E}(M) \cdot \mathfrak{E}(\mathfrak{N}))$ is a two-sided ideal in the associative algebra $\mathfrak{E}(G)$. By Lie's theorem on the simultaneous triangularizing of solvable linear Lie algebras (or by Engel's theorem) $\mathfrak{N}^k = 0$ where k is the dimension of the linear space on which G operates. Hence $\mathfrak{E}(\mathfrak{N})^k = (0)$ and $\mathfrak{L}(\mathfrak{E}(M) \cdot \mathfrak{E}(\mathfrak{N}))^k = \mathfrak{L}(\mathfrak{E}(M) \cdot \mathfrak{E}(\mathfrak{N})^k) = 0$. Set $S = \mathfrak{E}(M)$ and $\mathfrak{T} = \mathfrak{L}(\mathfrak{E}(M) \cdot \mathfrak{E}(\mathfrak{N}))$. $\mathfrak{T}$ is in the radical of $\mathfrak{E}(\mathfrak{G})$. Moreover M being fully reducible, $\mathfrak{E}(M)$ is fully reducible and is therefore a semi-simple associative algebra. Thus $\mathfrak{S} \cap \mathfrak{T} = (0)$, and $\mathfrak{E}(\mathfrak{G}) = \mathfrak{S} + \mathfrak{T}$ (semi-direct). That is, $\mathfrak{S} + \mathfrak{T}$ is a Wedderburn decomposition of $\mathfrak{E}(G)$. Thus assertion a) has been proved.

To prove b), we observe that $\mathfrak{T} \cap \mathfrak{G}$ is invariant under inner automorphisms $y \to gyg^{-1}$ with g in G and it is therefore invariant under the infinitesimal transformations $y \to [g, y]$ with g in the Lie algebra $\mathfrak{G}$. Thus $\mathfrak{T} \cap \mathfrak{G}$ is an ideal of the Lie algebra $\mathfrak{G}$. Since it consists of nilpotent elements, it is a nilpotent ideal and therefore contained in the radical of $\mathfrak{G}$ and hence $\mathfrak{T} \cap \mathfrak{G} \subset \mathfrak{N}$. But by definition of $\mathfrak{T}$, $\mathfrak{N} \subset \mathfrak{T} \cap \mathfrak{G}$. Therefore $\mathfrak{T} \cap \mathfrak{G} = \mathfrak{N}$.

To prove c), we recall that $u = \exp \log u$ for u in N. Hence $N \subset I + \mathfrak{E}(\mathfrak{N})$ and $(I + \mathfrak{T}) \cap G \supset N$. Conversely, $u \varepsilon (I + \mathfrak{T}) \cap G$ implies $\log u \varepsilon \mathfrak{T}$ and also $\log u \varepsilon \mathfrak{G}$, the latter since $\{\exp t \log u \mid t \varepsilon K\}$ is the smallest algebraic group containing u. Hence $\log u \varepsilon \mathfrak{T} \cap \mathfrak{G} = \mathfrak{N}$ and $u \varepsilon N$. Proof of the theorem is now complete. Analogously, one can prove

THEOREM. *Let $\mathfrak{G}$ be an algebraic Lie algebra and $\mathfrak{E}$ its associative enveloping algebra. Then there exists a Wedderburn decomposition $\mathfrak{S} + \mathfrak{T}$ for $\mathfrak{E}$ such that $\mathfrak{S} \cap \mathfrak{G}$ is a maximal fully reducible subalgebra of $\mathfrak{G}$ and $\mathfrak{T} \cap \mathfrak{G}$ is the ideal of nilpotent elements in the radical of $\mathfrak{G}$.*

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CORRECTIONS TO "PERIODIC MAPPINGS ON A BANACH ALGEBRA." *

By Bertram Yood.

Dr. H. Kamel has kindly pointed out that the proof of Lemma 4.1 part (2) of this paper (this Journal, vol. 77, pp. 17-28) is incorrect. We have been unable to supply a correct proof. Although this lemma is used in the subsequent arguments we show that all the conclusions of the paper mentioned in the introduction hold. Only some subsidiary results require modification.

Consider Theorem 4.6. The proof for $n=1$ on p. 25 is valid since T has period two. For typographical convenience set $H_{2^j}(K_{2^j})=H^j(K^j)$. We show by induction that $S \subset K^i$, $i=0,1,\cdots,n$ (whence $S=(0)$). This holds for $i=0$ by hypothesis and assume it for $0 \leqq i < n$. Set $V=T^{2^{i+1}}$. Let $\mathfrak{J}$ be any two-sided ideal of B where $\mathfrak{J} \subset K^i$. Since V is an automorphism, H^{i+1} is an algebra with $\mathfrak{J} \cap H^{i+1}$ as a two-sided ideal. By arguments of p. 25 given for J we have $\mathfrak{J} \cap H^{i+1}=(0)$. If $i+1=n$ then we already have $S \cap H^n=(0)$ and $S=(0)$. Suppose that $i+1<n$. Then

$$V(K^i)=(T^{2^i})^2(K^i)=K^i$$

by Lemma 2.1. Let r be the period of V. The algebraic sum

$$\mathfrak{J}=S+V(S)+\cdots+V^{r-1}(S)$$

is a two-sided ideal of B (since S is a two-sided ideal) and $\mathfrak{J} \subset K^i$ whence $\mathfrak{J} \cap H^{i+1}=(0)$. Let $y \in S$, $y=u+v$ in the decomposition

$$K^i=H^{i+1} \cap K^i \oplus K^{i+1}$$

of Lemma 2.2. Set $W=I+V+V^2+\cdots+V^{r-1}$. Then $W(y) \in \mathfrak{J}$, $W(v)=0$ and $W(u)=ru$. Thus $u=0$ and $y=v \in K^{i+1}$. This completes the induction.

This gives the correctness of all the results subsequent to Theorem 4.6 with the exception of Theorem 4.10. For that result we argue as follows. $\| x \|$ and $\| x \|_1 = \| T(x) \|$ are two complete norms on S. Let x_i be in separating ideal S_1 for these norms on S, and let $\| v_i - x_i \| \to 0$, $\| v_i \|_1 \to 0$ where $v_i \in S \subset K_1$, $i=1,2$. Arguing as in Lemma 4.9 we see that $x_1 x_2 = 0$. Thus

* Received August 30, 1955.

S_1 is a zero algebra. Now S is semi-simple being a two-sided ideal in B. Thus $S_1 = (0)$. By Theorem 2.4, S is a zero algebra whence $S = (0)$.

LEMMA. *Let T be a periodic automorphism or anti-automorphism on a Banach algebra B with separating ideal S. If T^2 is continuous then $T(S) = S$. If T^2 is continuous on $\bar{K}_1$, $u + v \varepsilon S$ where $u \varepsilon H_1$, $v \varepsilon K_1$, then $u \varepsilon S$ and $v \varepsilon S$.*

Suppose T^2 continuous. Let $y = u + v \varepsilon S$, $u_k + v_k \to y$, $u_k + T(v_k) \to 0$ where each u_k and $u(v_k$ and $v)$ lies in $H_1(K_1)$. Then $u_k + T^2 v_k \to u + T^2(v)$. Also $z_k = u - u_k + T(v - v_k) \to T(y)$ while $T(z_k) \to 0$. Thus $T(S) \subset S$ and by periodicity $T(S) = S$.

Suppose T^2 continuous on $\bar{K}_1$. Suppose first that the period n of T is even. Let $y \varepsilon S$ with the above notation. Then $w_k = v_k - v - T(v_k) \to u$ with $w_k \varepsilon K_1$ by Lemma 2.1. Then $u \varepsilon K_1' \cap H_1 \subset S$ by Lemma 4.2(b). Hence also $v \varepsilon S$. If n is odd then $T = T^{n+1}$ is continuous on $\bar{K}_1$. Now $v_k - T(v_k) \to u + v$. Operating by T on this $n - 1$ times and summing we obtain $nu = 0$. Thus $v \varepsilon S$.

The additional hypothesis $T(S) = S$ (or T^2 continuous) yields 4.1(3), (4), 4.2(a), 4.3 and 4.4(a). The additional hypothesis that T^2 is continuous on $\bar{K}_1$ yields 4.1(3), (4). For 4.4(b) assume $T(S) = S$ in case $n = 4$.

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CORRECTIONS TO THE PAPER "ENGEL RINGS AND A RESULT OF HERSTEIN AND KAPLANSKY."*

By M. P. Drazin.

In the above-named paper (this Journal, vol. 77 (1955), pp. 895-913), Theorem 6.4 should have been stated only for rings of zero characteristic: the argument for the case of prime characteristic breaks down in the last formula line on p. 911. This involves jettisoning Theorem 6.5 (the implied "proof" of which depended on applying Theorem 6.4 to homomorphs which cannot be guaranteed to have zero characteristic, even when the given ring has).

However, the writer has no evidence that Theorem 6.4 as stated or Theorem 6.5 is actually false. And in any case, since the valid part of the proof of Theorem 6.4 establishes the weaker conclusion $[x^m, y^{np^r}] = 0$ *without* characteristic hypothesis, the remarks following Theorem 6.4 still hold good, provided that we modify the final parenthetical clause so as to read "which would at any rate imply that every weak K-ring R with k, m, n satisfying (a) or (b) has R/J commutative."

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* Received December 6, 1955.